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Random-Time Aggregation in Partial Adjustment Models

Abstract

How is econometric analysis (of partial adjustment models) affected by the fact that, while data collection is done at regular, fixed intervals of time, economic decisions are made at random intervals of time? This paper addresses this question by modelling the economic decision making-process as a general point process. Under *random-time aggregation*: (1) inference on the speed of adjustment is biased - adjustments are a function of the *intensity* of the point process and the proportion of adjustment; (2) inference on the correlation with exogenous variables is generally downward biased; and (3) a non-constant intensity of the point process gives rise to a general class of regime dependent time series models. An empirical application to test the production-smoothing-buffer-stock model of inventory behavior illustrates, in practice, the effects of random-time aggregation.

- *Key words:* Point process; Intensity; Speed of Adjustment; S,s rules.
- *JEL classification:* C51; C43; C81

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1 Introduction

This paper deals with the observation that, while data collection is done at regular fixed intervals of time (every month, quarter, year, etc.), agents make decisions at irregular intervals of time. For example, in the stock market, the time between transactions often varies from less than a second to more than an hour. This statement pertains to many situations in economics and is particularly relevant in Partial Adjustment Models (P.A.M.), a popular model in economics used to describe inventory behavior, investment, short-run changes in employment, pricing policies, stock of money corrections, and other economic phenomena. In recent work by Caballero and Engel (1993, 1994), Caballero, Engel, and Haltiwanger (1994), macroeconomic relations with P.A.M. are modelled with close attention to microeconomic behavior and the effects of aggregation across heterogeneous agents. However, the effects of the timing and duration of decision making on statistical inference and econometric modelling are generally overlooked. It is crucial to understand that data would need to be disaggregated not only by agent, but by *decision*.

The usual approach in the analysis of temporal aggregation is to proceed under the assumption that agents make decisions at fixed intervals of time. However, the assumption that this interval coincides with the data sampling interval is abandoned. The goal of this type of analysis is to evaluate the consequences of the specification error that results when the agent's true decision interval is finer than the data sampling interval. Christiano and Eichenbaum (1987) term this "Temporal Aggregation Bias."

This paper takes this discussion one step further. It is concerned with the specification error that results when the agent's decision interval is *random* and, therefore, does not coincide with the data sampling interval: this phenomenon is termed "Random-Time Aggregation." More specifically, this paper assumes that the economic decision process can be well described by a general point process.

The following highlights the basic findings of this paper. When the decision-making process is explicitly modelled, the correct specification of the speed of adjustment is shown to depend on the intensity of the decision process and the proportion of adjustment. The resulting values are compatible with a wide range of models of adjustment. This explains a basic puzzle in the empirical literature of partial adjustment models: the implausibly low speeds of adjustment that are estimated. While this is especially true in models of inventory behavior (such as Blinder 1981, 1986; Blinder and Maccini 1991 and Feldstein and Auerbach 1976), it is also the case with other variables such as employment and investment (see for example Sims 1974; Phipps 1975; Smyth 1986; Nickell 1986 and Hamermesh 1989). As a result, in general, it is not possible to distinguish a convex from a non-convex cost of adjustment structure with the usual methods.

In the same spirit, random-time aggregation affects correlations and causal relations. The downward biases that are introduced result in the failure to detect these relationships in practice. This result is in marked contrast to the conjectures advanced in Christiano and Eichenbaum (1987). The framework is then generalized to allow for non-constant intensities of decision making. This generalization gives rise to a class of regime dependent models and yields a natural explanation for why errors in the model present a type of ARCH structure.

The last section of the paper provides an empirical application to glass containers inventories and shipments. The popular production smoothing buffer-stock model is tested on physical, seasonally unadjusted data. A non-linear model designed to partially account for random time aggregation obtains speeds of adjustment compatible with the predictions of the theory.

The paper is organized as follows: Section 2 introduces the notions of a counting process and a point process; Section 3 introduces the benchmark model for our discussion – a general partial adjustment model; Section 4 shows how theoretical models of adjustment give rise to Poisson point processes; Section 5 considers general models of inhomogeneous Poisson point processes; Section 6 suggests how to detect random-time aggregation in practice; Section 7 presents an empirical application; and Section 8 concludes and presents areas of future research.

2 Preliminary Concepts

This section introduces the concepts of a random point process, a counting process, and a *Poisson* process. A large amount of literature is dedicated to the study of these subjects with a high level of mathematical detail and rigor. Andersen, Borgan, Gill, and Keiding (1991) provide an in-depth analysis of counting processes. Snyder and Miller (1992) provide a good introductory analysis of point processes and their applications. The aim of this section is to provide an intuitive and operational understanding of the main ideas. Much of the discussion that follows is based on Snyder and Miller (1991).

A random point process is a mathematical model for a phenomenon (in this case, decision making) characterized by highly localized events (decisions), distributed randomly in time (in greater generality, a continuum space; time being an example of such a space). In the model, each event is represented by an idealized point representing the time at which the decision was made. A realization of a random point process on time is a set of points-in-time. In practice, it is often of interest to count the number of points in an interval of time on which a point process is defined. A counting process is introduced for this purpose and can be associated with every point process.

Definition 1 . Counting process: *Consider the process of counting events on some interval of time* $[s, u)$ *where* $\{t : s \leq t < u\}$ *and* $t_0 \leq s$ *. Then* $N(\{t : s \leq t < u\}) \equiv N(s, u)$ *is defined as the number of points in the interval from* s *to* u*, possibly including a point in* s *but not at* u*. We introduce the stochastic process* $\{N(t) : t \geq t_0\}$ *according to the definition:* $N(t) \equiv N(t_0) + N(t_0, t)$ *where* $N(t_0)$ *is a non-negative integer that we will set to zero for convenience. We will call* $N(t)$ *a counting process.*

Within the class of counting processes, we assume that the counting process is *Poisson*. Formal discussions of the conditions that a counting process has to meet in order for it to be Poisson and micro-economic models of adjustment that produce Poisson decision making processes are delayed to Section 4. Here, the main properties of the Poisson are stated.

Definition 2 . Intensity*: Define* $\lambda(t)$ *as the instantaneous average rate that points occur at time* t*. This is termed as the intensity function. When* $\lambda(t)$ *is a constant, independent of time, the corresponding Poisson process is said to be homogeneous, otherwise, it is termed inhomogeneous. Then:*

1. $\Pr[N(t) = 0] = \exp\left\{-\int_0^t \lambda(u)du\right\}$

2. $\Pr[N(t)=n]=\frac{1}{n!}\left(\int_0^t \lambda(u)du\right)^n \exp\left\{-\int_0^t \lambda(u)du\right\}$

3 A Model of Adjustment

This section analyzes the effects of random-time aggregation in partial adjustment models, relying on a reduced form model which is not derived here from first principles. Rotemberg (1987) shows the equivalence of our formulation to a linear/quadratic model. Caballero and Engel (1994) use it as an encompassing framework for the linear/quadratic and (S, s) economies.

The following introduces the notation used in this paper. The subscript t in a variable is the usual time index and denotes the value of that variable in the sampling interval t. Let $\Delta_t \equiv t-(t-1)$ then Δ_t is a constant equivalent to the length of the sampling interval (e.g. for quarterly data it is three months). Next, consider the *event-time* index τ formally defined as $\tau_i(r,u)\equiv\tau_i$ which denotes the date within the interval $[r,u)$ when event $i=1,2,...,I$ takes place. In this paper, each event is equivalent to a *decision* made by an economic agent. At times, the subscript i will be dropped in which case it will be understood that $\tau-1$ refers to the date of the decision immediately prior to the subsequent decision made at time τ. Under the assumption that the arrival of decisions per unit time is Poisson distributed with intensity λ, then $\Delta_\tau \equiv \tau-(\tau-1)$ is an exponentially distributed random variable with mean $1/\lambda$.

Define y^* as the *target* variable and y as the *control* variable. The agent can act upon y but not on y^*. The agent's objective is to keep the distance $|y^*_\tau - y_{\tau-1}|$ as small as possible subject to a cost of adjustment. Depending on each particular situation, the agent might be penalized by a fixed amount per decision. Additionally, there may be costs proportional to the size of the adjustment. These costs can be asymmetric in nature: positive adjustments may bear different costs than negative adjustments. For instance, consider a firm that tries to keep its labor force as close as possible to the optimal level. Since there are costs of changing the labor force (firing costs such as severance payments; hiring expenses such as screening and interviewing of new candidates; etc.), changes in the firm's employment level do not happen continuously, but rather at irregular intervals.

The desired level y^* will depend on several factors linked to expectations on the cost structure of the problem at hand. These factors will typically include expectations on real wages, real interest rates, prices of raw materials, future sales, prices, etc. The process for y^* in general, can be expressed as the following Itô process:

$$dy^*(t) = \mu[y^*(t), w(t), t]dt + \sigma[y^*(t), w(t), t]d\xi(t) \tag{1}$$

where $w(t)$ is a vector of variables that determine $y^*; \xi(t)$ is assumed to be Gaussian with increments having mean 0 and covariance $E[d\xi(t), d\xi(t')] = dt$ if $t = t'$ and equal 0 for $t \neq t'$ (for a precise interpretation see Bergstrom 1984). In what follows, it will be assumed that $\mu[y^*(t), w(t), t] = \mu$ and $\sigma[y^*(t), w(t), t] = \sigma$. This simplification makes the exposition of the central points clear but does not represent a fundamental assumption of the problem and can be relaxed in favor of richer expressions. Consequently, the stochastic process for the control, y is simply:

$$y_\tau = y_{\tau-1} + \alpha(y^*_\tau - y_{\tau-1}) \tag{2}$$

where $\alpha \in [0, 1]$ and will be assumed constant throughout. Again, one can always complicate the analysis with asymmetries, non-constant expressions, etc. The parameter α is termed as the *speed of adjustment* parameter. Expression 2 is specified in terms of the event-time index τ.

If $\alpha = 1$ then there are only costs associated with making a decision, regardless of the size of the disequilibrium $|y^*_\tau - y_{\tau-1}|$. Traditionally, this corresponds to the framework found in menu-cost models and (S, s) economies. Linear/quadratic models are commonly associated with $0 < \alpha < 1$. Caballero and Engel (1994) consider the cases where α is a linear and then a quadratic function of $y^*_\tau - y_{\tau-1}$.

Next define $z_\tau \equiv y^*_\tau - y_{\tau-1}$. Using equations 1 and 2 it is easy to see that

$$z_\tau = \left[\mu\Delta_\tau + y^*_{\tau-1} + \sigma \int_{\tau-1}^{\tau} d\xi(r)\right] - y_{\tau-1} \tag{3}$$

From 2

$$y_{\tau-1} = y_{\tau-2} + \alpha(y^*_{\tau-1} - y_{\tau-2}) \tag{4}$$

combining 3 and 4, the law of motion for z becomes:

$$z_\tau = \mu\Delta_\tau + (1-\alpha)z_{\tau-1} + \sigma \int_{\tau-1}^{\tau} d\xi(r) \tag{5}$$

The focus of the analysis consists in estimating the parameter α.

3.1 The Adjustment Process

This section describes the adjustment process under the assumption that decision making follows a counting process well described by the Poisson. Let n be the number of decisions (adjustments) made per sampling interval $[t-1,t)$, then:

$$P_n \equiv \Pr[N(t-1,t)=n] = \frac{1}{n!}\left(\int_{t-1}^{t}\lambda(u)du\right)^n \exp\left\{\int_{t-1}^{t}\lambda(u)du\right\} \tag{6}$$

For the sake of clarity, consider a homogeneous Poisson process with intensity $\lambda(t)=\lambda$. λ is the average number of decisions per sampling interval. If we were to consider quarterly data, 4λ would be the average decision rate in a year and $\frac{1}{3}\lambda$ in a month.

Now let $\tau_i(t-1,t)=\tau_i$ denote the time within the interval $[t-1,t)$ at which decision i is made. By convention, we assume $\tau_0=t-1$. Therefore, $\eta_i \equiv \tau_i - \tau_{i-1}$ for $i \geq 1$ is the duration of time between decisions in the interval $[t-1,t)$: a random variable. For a homogeneous Poisson process, $\eta_i \sim \mathcal{E}(\lambda)$, where $\mathcal{E}(\lambda)$ is the exponential distribution with parameter λ.

Consider the mechanics of the adjustment process expressed in terms of observables. If the number of decisions per sampling interval $[t-1,t)$ is:

- none: $z_t = z_{t-1} + \mu + \sigma \int_{t-1}^{t} d\xi(r)$;
- one: $z_t = (1-\alpha)\left[z_{t-1} + \mu\eta_1 + \sigma\int_{t-1}^{\tau_1} d\xi(r)\right] + (1-\eta_1)\mu + \sigma\int_{\tau_1}^{t} d\xi(r)$;
- two: $z_t = (1-\alpha)\left\{(1-\alpha)\left[z_{t-1} + \mu\eta_1 + \sigma\int_{t-1}^{\tau_1} d\xi(r)\right] + \mu\eta_2\right.$
  $\left.+\sigma\int_{\tau_1}^{\tau_2} d\xi(r)\right\} + (1-\eta_1-\eta_2)\mu + \sigma\int_{\tau_2}^{t} d\xi(r)$;
- and so on.

The intuition behind this process is the following: When no adjustments are made in the interval $[t-1,t)$, z_t (the disequilibrium variable) attains the same value as at the beginning of the period $(t-1)$ plus the disequilibrium accumulated in the interval. This disequilibrium is a reflection of how much the target has "travelled" since (i.e., $\mu+\sigma\int_{t-1}^{t} d\xi(r)$). When one adjustment takes place sometime within $[t-1,t)$, the correction is made on the disequilibrium existing at the time of adjustment. This is the sum of the inherited disequilibrium, $z_{t-1,}$ plus whatever the target has travelled by date τ_1. This is given by $\mu\eta_1+\sigma\int_{t-1}^{\tau_1} d\xi(r)$. From this explanation, it is easy to see how we proceed with two, three, or more adjustments.

In general, the process for z_t can be described as follows:

$$
\begin{aligned}
z_t \quad = \quad & (1-\alpha)^{k_t} z_{t-1} + \left\{ \mu \sum_{i=1}^{k_t} (1-\alpha)^i \eta_i + \mu \left[1 - \sum_{i=1}^{k_t} \eta_i \right] \right\} + \\
& \left\{ \sum_{i=1}^{k_t} (1-\alpha)^i \varepsilon_i + \varepsilon_{k_t+1} \right\}
\end{aligned} \tag{7}
$$

where k_t is the number of adjustments in the interval $[t-1,t)$. $k=\{k_t\}_{t=1}^{T}$ is a stochastic process hereby assumed to be Poisson distributed. $\varepsilon_i = \sigma\int_{\tau_{i-1}}^{\tau_i} d\xi(r), \varepsilon_{k_t+1} = \int_{\tau_{k_t}}^{t} d\xi(r)$, therefore, $\varepsilon_i \sim N(0,\sigma^2\eta_i)$. In other words, the mean (given by the first term in brackets) and the autoregressive coefficients depend on k_t which varies for each interval considered. The error term, the second term in brackets, is the sum of non-identical, independent, normally distributed random variables: a normal random variable with zero mean, whose variance also depends on k_t. Note that k_t is typically unobserved.

In general, we would like to model the joint probability distribution of z_t and k_t, conditional on the past. Let $\mathcal{Y}_t$ denote the history of observations through date t

$$\mathcal{Y}_t = \{z_t, k_t, z_{t-1}, k_{t-1}, ..., z_1, k_1\}$$

then, without loss of generality, this probability can be factored as the product of the conditional distribution times the marginal distribution,

$$f(z_t, k_t|\mathcal{Y}_{t-1}; \theta_1, \theta_2) = g(z_t|k_t, \mathcal{Y}_{t-1}; \theta_1) q(k_t|\mathcal{Y}_{t-1}; \theta_2) \tag{8}$$

However, partial adjustment models are usually specified as:

$$z_t = \rho_0 + \rho_1 z_{t-1} + \varepsilon_t \tag{9}$$

The implicit assumption here (at a minimum) is that $k_t = k\ \forall t$ and therefore the marginal distribution of k_t collapses to 1. Estimation of Equation 9 by maximum likelihood say, is therefore equivalent to estimating $E_{t-1}(z_t)$ unconditionally with respect to k_t. Under the assumption that the k_t are Poisson distributed, this expectation becomes,

$$\begin{aligned} E_{t-1}(z_t) &= P_0 \left[z_{t-1} + \mu + \sigma \int_{t-1}^{t} d\xi(r) \right] + \\ &+ P_1 \left[(1-\alpha) \left[z_{t-1} + \mu\eta_1 + \sigma \int_{t-1}^{\tau_1} d\xi(r) \right] + (1-\eta_1)\mu + \sigma \int_{\tau_1}^{t} d\xi(r) \right] + \\ &+ P_2 \left[\begin{array}{c} (1-\alpha)\{(1-\alpha) \left[z_{t-1} + \mu\eta_1 + \sigma \int_{t-1}^{\tau_1} d\xi(r) \right] + \mu\eta_2 + \\ + \sigma \int_{\tau_1}^{\tau_2} d\xi(r)\} + (1-\eta_1-\eta_2)\mu + \sigma \int_{\tau_2}^{t} d\xi(r) \end{array} \right] + ... \end{aligned} \tag{10}$$

with $P_i = Pr[k_t = i|\mathcal{Y}_{t-1}]$. After some algebra reported in the Appendix, it can be shown that

$$E_{t-1}(z_t) = z_{t-1} \exp\left\{-\alpha \int_{t-1}^{t} \lambda(u)du\right\} + \overline{\mu} \tag{11}$$

with:

$$\begin{aligned} \overline{\mu} &= \mu \exp\left\{-\alpha \int_{t-1}^{t} \lambda(u)du\right\} \\ &\left(\sum_{i=0}^{\infty} \frac{\left(\int_{t-1}^{t} \lambda(u)du\right)^i}{i!} \left[\sum_{j=1}^{i} (1-\alpha)^j \overline{\eta}_{i+j+1} + 1 - \sum_{j=1}^{i} \overline{\eta}_j\right]\right) \end{aligned} \tag{12}$$

where $\overline{\eta}_j = E_{t-1}(\eta_j)$. Empirically, the coefficient of interest is that associated with z_{t-1}. From Equations 9 and 11 and assuming a constant intensity $\lambda(t) = \lambda$, it is easy to see that $\widehat{\rho}_1 = \exp\{-\alpha\lambda\}$. Therefore, when assuming $k_t = k = 1\ \forall t$, the interpretation given to $\widehat{\rho}_1$ is that of an estimate of $(1-\alpha)$ which in light of Equation 11 is clearly incorrect. The bias in the structural interpretation of the adjustment parameter becomes,

$$BIAS_\alpha = 1 - \alpha_0 - exp\left(-\alpha_0 \int_{t-1}^{t} \lambda(u)du\right) \tag{13}$$

where α_0 denotes the true value of α. The importance of this bias is best illustrated with an example. Let $\lambda = 1$ with monthly data. This is the best case scenario, where the frequency with which our data is sampled coincides with the average frequency of decision making. Assume $\alpha_0 = 1$ (non-convex costs of adjustment); then, $\widehat{\rho}_1 \simeq 0.37$. Under the traditional assumptions, this is interpreted as $\widehat{\alpha} = 0.63$, thus incorrectly concluding that it takes 3 months to close 95% of any given disequilibrium. However, note that the real speed of adjustment is 100% per month on average, i.e. more than three times faster. As the frequency of adjustment (λ) increases, the bias increases up to $(1-\alpha_0)$. Conversely, infrequent adjustments work in the opposite direction and the bias can achieve a value of $-\alpha_0$. Figure 1 displays the shape of this bias as a function of λ.

These results do not arise from using data sampled at a lower frequency than the decision-making process, but rather from ignoring the temporal pattern of decision making altogether. As a consequence, inference on speed of adjustments can be severely incorrect. In addition, testing an (S,s) model against an alternative linear/quadratic model (for example) becomes complicated unless one can jointly model adjustments and decisions. Therefore, rather than changing the sampling frequency of the data, ideally one would want data sampled per decision.

3.2 A Link to Continuous-Time Models

It is instructive to compare the results in the previous section to the literature on time deformation introduced by Stock (1988). Stock's original formulation considers a time series model based on an "economic" time-scale and a time-scale transformation of the process into an observed, calendar-based time-scale. The basic ingredients of his framework consist of a generic continuous time model and a "latent" process that evolves in operational time τ. The process is observed in calendar time t however, which is related to the operational time-scale by a continuous transformation $\tau = \varphi(t)$. Now recall Equation 5, the basis of our discussion,

$$z_\tau = \mu\Delta_\tau + (1-\alpha)z_{\tau-1} + \sigma \int_{\tau-1}^{\tau} d\xi(r)$$

Its continuous time counterpart can be rewritten as the following Ornstein-Uhlenbeck process,

$$dz(\tau) = (\mu - \alpha z(\tau))\, d\tau + \sigma d\xi(\tau) \tag{14}$$

which has the following discrete-time representation (see Bergstrom (1984)),

$$\begin{aligned} z(\tau) &= (1 - exp(-\alpha\Delta_\tau))\,\frac{\mu}{2} + exp\,(-\alpha\Delta_\tau)\, z(\tau-1) + \\ &\quad \sigma \int_{\tau-1}^{\tau} exp(-\alpha(\tau - r))d\xi(r) \end{aligned} \tag{15}$$

Let $\tau = \varphi(t)$ and $\tau - 1 = \varphi(t-1)$, then the observable process in discrete time becomes,

$$\begin{aligned} z_t &= (1 - exp(-\alpha\Delta\varphi(t)))\,\frac{\mu}{2} + exp\,(-\alpha\Delta\varphi(t))\, z(\tau-1) + \varepsilon_t \\ \varepsilon_t &= \sigma \int_{\varphi(t-1)}^{\varphi(t)} exp(-\alpha(\varphi(t) - r))d\xi(r) \end{aligned} \tag{16}$$

Equation 16 bears close similarity to 11 and 12. The model will generally exhibit time-varying covariates and heteroskedasticity that depend on the time-scale transformation. The frequency of decision-making per time interval $(t-1, t]$ is a natural candidate for the time-scale transformation $\Delta\varphi(t)$. After all, economic-time will depend on how often agents make economic decisions. Accordingly, setting $\Delta\varphi(t) = \int_{t-1}^{t} \lambda\,(r)\, dr$, 16 can be rewritten as,

$$z_t = \left(1 - exp(-\alpha \int_{t-1}^{t} \lambda(r)dr)\right) \frac{\mu}{2} + exp\left(-\alpha \int_{t-1}^{t} \lambda(r)dr\right) z(\tau-1) + \varepsilon_t \tag{17}$$

where $E(\varepsilon_t) = 0$; $E(\varepsilon_t^2) = \sigma \left(exp\left(2\alpha \int_{t-1}^{t} \lambda(r)dr\right) - 1\right)/2\alpha$. Therefore, an adjustment process subject to random time aggregation yields the same adjustment parameter function as a continuous time adjustment process subject to time deformation where the time-scale transformation is given by the intensity of economic decision-making, $\int_{t-1}^{t} \lambda(r)dr$.

3.3 Including Exogenous Variables

Although Equation 5 is a useful characterization of the dynamics of disequilibria and it has been used in recent papers by, for example, Caballero and Engel (1994), Caballero, Engel, and Haltiwanger (1994), two modifications of the previous analysis are considered. First, we analyze the effect of random-time aggregation on the inference of correlations and/or causal relations between variables. Second, we will introduce this generalization in the context of another popular specification of the P.A.M., alternative to that used in Equation 5.

Based on Equation 4, the partial adjustment model is sometimes specified as,

$$\Delta y_\tau = \alpha \left(y^*_\tau - y_{\tau-1} \right) + \beta w_\tau \tag{18}$$

Here α retains the same interpretation as before and β captures the effect of w on changes in y. A good illustration of such a model is the production-smoothing model of inventories:

$$\Delta I_{t+1} = a \left(I^*_t - I_t \right) - b(S_t - S^e_t) \tag{19}$$

where I_t is the stock of inventories; I^*_t is the desired level of inventories and $(S_t - S^e_t)$ are sales surprises. Accordingly, inventories change for two motives. The first corresponds to the term $(I^*_t - I_t)$ and reflects anticipated inventory investment. Desired inventories will typically depend on expected sales, expected costs and current and expected real interest rates. The second term, $(S_t - S^e_t)$, corresponds to unanticipated inventory investment and captures the extent to which inventories buffer sales surprises to maintain production approximately constant. The empirical section of this paper will be based on this model.

Assume that w follows a general continuous-time Markov process. Like before, one could think of w following a generic p^{th} order stochastic differential equation. However, cumbersome derivations would detract from the central argument. Consequently, let,

$$dw(t) = mdt + d\zeta(t) \tag{20}$$

where $\zeta(t)$ is assumed Gaussian with increments having mean 0 and covariance $E[d\zeta(t), d\zeta(t')] = \sigma_w dt$ for $t = t'$ and equal 0 for $t \neq t'$. Furthermore, $E[d\zeta(t), d\xi(t')] = 0$ for all t and t'. Under these assumptions, consider the adjustment process in terms of sampled observations. If the number of decisions made per interval of time $(t-1, t]$ is:

- none: $\Delta y_t = -\beta\left(w_{t-1} + m + \int_{t-1}^{t} d\zeta(r)\right)$;
- one: $\Delta y_t = \alpha\left[z_{t-1} + \mu\eta_1 + \int_{t-1}^{\tau_1} d\xi(t) - \beta\left(w_{t-1} + m\eta_1 + \int_{t-1}^{\tau_1} d\zeta(t)\right)\right] -$
  $\beta\left(m(1-\eta_1) + \int_{\tau_1}^{t} d\zeta(t)\right)$;
- two: $\Delta y_t = \alpha\left\{\begin{array}{c}(1-\alpha)\left[z_{t-1} + \mu\eta_1 + \int_{t-1}^{\tau_1} d\xi(t) - \beta\left(w_{t-1} + m\eta_1 + \int_{t-1}^{\tau_1} d\zeta(t)\right)\right] + \\ +\mu\eta_2 + \int_{\tau_1}^{\tau_2} d\xi(t) - \beta\left(m\eta_2 + \int_{\tau_1}^{\tau_2} d\zeta(t)\right)\end{array}\right\} -$
  $-\beta\left(m(1-\eta_1-\eta_2) + \int_{\tau_2}^{t} d\zeta(t)\right)$;
- and so on.

Recall $\eta_i = (\tau_i - \tau_{i-1})$. The intuition behind this adjustment process is clear. When no adjustments are made during the sampling interval $[t-1, t)$, y will change due to the evolution of w from $t-1$ to t. If an adjustment takes place during the interval, the disequilibrium existing up to that point will be corrected. However, now the disequilibrium not only depends on movements on the desired level y^* but also on the impact of w. When two adjustments are made, the second correction is with respect to the proportion $(1-\alpha)$ of the first correction plus any additional movements in z and w since and up to τ_2. It is easy to see how two or more adjustments would be done. Rewriting the observed adjustment process in a similar way to Equation 7,

$$\Delta y_t = \gamma(k_t) + \delta\left(k_t\right)\alpha(1-\alpha)^{k_t-\delta(k_t)} z_{t-1} + \beta\alpha^{\delta(k_t)}(1-\alpha)^{k_t-\delta(k_t)} w_{t-1} + u_t \tag{21}$$

where $k = \{k_t\}_{t=1}^{t=T}$ is the same Poisson distributed stochastic process previously discussed; $\delta(k_t) = 1$ if $k_t > 0$ and equals 0 otherwise. The intercept $\gamma(k_t)$ is,

$$\gamma(k_t) = \mu \sum_{i=1}^{k_t} \alpha(1-\alpha)^{i-1}\eta_i + \beta m \left(\sum_{i=1}^{k_t} \alpha(1-\alpha)^{i-1}\eta_i + \left[1 - \sum_{i=0}^{k_t} \eta_i \right] \right) \tag{22}$$

where $\eta_0 = 0$, and the error term is,

$$u_t = \sum_{i=1}^{k_t} \alpha(1-\alpha)^{i-1} \int_{\tau_{i-1}}^{\tau_i} d\xi(r) + \beta \left(\sum_{i=1}^{k_t} \alpha(1-\alpha)^{i-1} \int_{\tau_{i-1}}^{\tau_i} d\zeta(r) + \int_{\tau_{k_t}}^{t} d\zeta(r) \right) \tag{23}$$

where $\tau_0 = t$. Despite the apparent complexity of the above equations, 21 is simply a linear equation with time-varying parameters that depend on k_t. The error term is the sum of non-identical, independently distributed normal random variables with mean zero and variance that depends on k_t, i.e. $u_t \sim N(0, \sigma_u(k_t))$. As k_t increases, the coefficients associated with z_{t-1} and w_{t-1} will decrease toward zero, provided that $\alpha < 1$. If $\alpha = 1$ then the coefficient associated with z_{t-1} is zero except for $k_t = 1$ when it equals one. The coefficient associated with w_{t-1} is β if $k_t = 0$, is $\beta\alpha$ if $k_t = 1$ and is zero otherwise. The effect on w_{t-1} is particularly interesting since it stems, not from time aggregation per se, but rather from the mechanics of adjustment under random-time aggregation.

In applied work, it is common to specify Equation 21 as,

$$\Delta y_t = \phi_0 + \phi_1 z_{t-1} + \phi_2 w_{t-1} + v_t \tag{24}$$

For instance, Blanchard (1983) reports estimates of ϕ_1 that range from 0.27 to 0.02 for monthly automobile inventory data. With quarterly data, Feldstein and Auerbach (1976) report $\phi_1 = 0.057$ for manufacturing, nondurable inventory data. Blinder (1981) estimates similar specifications for several durable and non-durable manufacturing industries finding values of ϕ_1 between 0.14 to 0.03. Estimates of "speeds of adjustment" in the labor literature are similar. Sargent (1978) and Meese (1980) report $\phi_1 = 0.05$ for quarterly U.S. unemployment. For the U.K., Nickell (1984) finds $\phi_1 = 0.15$. Abraham and Housman (1993) obtain $\phi_1 \in [0.06, 0.19]$ for Germany, France and Belgium. Recent efforts by Hamermesh (1993) with micro-data for U.S. Manufacturing yield similar results, $\phi_1 = 0.16$.

Maximum likelihood estimation of Equation 24 implicitly assumes that $k_t = k$ (more specifically $k = 1$). Consequently, in the same spirit of our discussion in Section 3.1, this is equivalent to calculating $E_{t-1}(\Delta y_t)$ unconditionally with respect to the distribution of k_t. Under our maintained assumptions, this expectation becomes,

$$\begin{aligned} E_{t-1}(\Delta y_t) &= c + \frac{\alpha}{(1-\alpha)}\left[exp\left\{-\alpha\int_{t-1}^{t}\lambda(u)du\right\} - exp\left\{-\int_{t-1}^{t}\lambda(u)du\right\}\right] z_{t-1} \qquad (25) \\ &+ \frac{\beta}{(1-\alpha)}\left[\alpha exp\left\{-\alpha\int_{t-1}^{t}\lambda(u)du\right\} + (1-2\alpha)exp\left\{-\int_{t-1}^{t}\lambda(u)du\right\}\right] w_{t-1} \end{aligned}$$

Accordingly, $\widehat{\phi}_1 = [\alpha/(1-\alpha)]\,[exp(-\alpha\lambda) - exp(-\lambda)]$ (for a homogeneous Poisson) and $\widehat{\phi}_2 = [\beta/(1-\alpha)]\,[\alpha exp(-\alpha\lambda) + (1-2\alpha)exp(-\lambda)]$. However, these coefficients are typically interpreted as $\widehat{\phi}_1 = \alpha$ and $\widehat{\phi}_2 = -\beta$. It is easy to calculate their biases,

$$\begin{aligned} BIAS_\alpha &= \frac{\alpha_0}{(1-\alpha_0)}\left\{exp(-\alpha_0\lambda) - exp(-\lambda)\right\} - \alpha_0 \\ BIAS_\beta &= \frac{\beta_0}{(1-\alpha_0)}\left\{\alpha_0 exp(-\alpha_0\lambda) + (1-2\alpha_0)exp(-\lambda)\right\} - \beta_0 \end{aligned} \tag{26}$$

where α_0 and β_0 denote the true values of α and β. An example best illustrates the effects of these biases. Consider Feldstein and Auerbach's (1976) example, which used quarterly data. They report estimates of $\widehat{\phi}_1 = 0.057$ and $\widehat{\phi}_2 = 0.044$ with inventory data. For $\lambda = 4.5$ (a little over one adjustment per month) the speed of adjustment becomes $\alpha = 0.85$ and the true effect of sales on inventories $\beta = 0.61$. As λ grows, then $BIAS_\alpha$ tends to $-\alpha_0$ and $BIAS_\beta$ to $-\beta_0$. Conversely, as λ tends to 0, $BIAS_\alpha$ tends to $-\alpha_0$ and $BIAS_\beta$ to 0. Figure 2 illustrates the shape of these biases as a function of λ.

4 The Poisson Point Process and the Law of Rare Events

This section justifies the generality of assuming the Poisson distribution in modeling decision-making processes by stating the Law of Rare Events (L.R.E.). As an illustration, the generating process of decisions for an (S,s) rule is considered – an adjustment rule that naturally produces Poisson distributed decision making. The Law of Rare Events is a set of qualitative conditions for an arbitrary counting process to be a Poisson counting process. These conditions are stated in an intuitive way. For the formal mathematical statement of the theorem and its proof, the reader is referred to Snyder and Miller (1991). Lancaster (1992) and Taylor and Karlin (1994) provide alternative statements although with less generality.

Theorem 3 . The Law of Rare Events: *Let $\{N(t) : t \geq t_0\}$ be a counting process associated with a point process on $[t_0, \infty)$. Suppose that:*

1. *The point process is conditionally orderly. Informally, this means that points do not occur simultaneously in time.*

2. *For all $t \geq t_0$ and for an arbitrary event P associated with the random variables $\{N(u) : t_0 \leq u \leq t\}$, the limit*

$$\lim_{\delta \to 0} \frac{\Pr[N(t, t+\delta) = 1|P]}{\delta} = \lambda(t)$$

   exists, is finite, and is an integrable function of t alone, that is, $\int_s^t \lambda(u)du$ exists and is finite for all finite intervals $[s, t]$, $t_0 \leq s \leq t$.

3. $\Pr[N(t_0) = 0] = 1$.

Then $\{N(t) : t \geq t_0\}$ is a Poisson counting process with an absolutely continuous parameter function $\Lambda(t) = \int_{t_0}^t \lambda(u)du$ and

$$\Pr[N(s,t) = n] = \frac{1}{n!}\left(\int_s^t \lambda(u)du\right)^n \exp\left\{-\int_s^t \lambda(u)du\right\}$$

Hypothesis 2 implies that the process $\{N(t) : t \geq t_0\}$ evolves infinitesimally without after-effects: If P and F are arbitrary events in the past and in the future respectively, the conditional probability of F given P, equals the unconditional probability of P for all $t \geq t$. In other words, the process $\{N(t) : t \geq t_0\}$ has independent increments. The function $\lambda(t)$

is the instantaneous average rate at which points occur and is termed the *intensity function.* When the intensity is a constant independent of time, $\lambda(t) = \lambda$, the corresponding Poisson counting process is said to be *homogeneous.* In this case:

$$\Pr[N(t) = n] = \frac{1}{n!}\lambda^n e^{-\lambda}$$

The application of the L.R.E. is best illustrated in the context of (S,s) rules which are constructed from formal microeconomic models of discontinuous and lumpy adjustment and are applied in a variety of topics in economics. While their origin can be traced back to the theory of inventory control, today they are also used to describe investment, cash balance adjustments, labor demand, technology upgrades, etc. Optimality of such rules is not proved here but rather the temporal pattern of adjustments is analyzed.

There are several types of (S,s) rules: one sided, double sided, with double return points, etc. This section analyzes a one-sided (S,s) rule, described as follows: Let $s < S$ and let $y^*(t)$ be a Brownian motion with drift $\mu \geq 0$, where $y^*(0) = s$. Whenever $y^*(t) \geq S$, the process is restarted at s, that is, $dy^*(t_S) = S - s$, where t_S is the first time the process reaches the level S. Then, it can be shown that t_S has probability density function:

$$f(t, s, S) = \frac{(S-s)}{\sigma\sqrt{2\pi t^3}} \exp\left[\frac{((S-s) - \mu t)^2}{2\sigma^2 t}\right] \qquad t > 0 \tag{27}$$

Proof: Karlin and Taylor (1975), p. 363.

Equation 27 is the inverse gaussian distribution and provides the waiting time density until the first decision. Note that the process restarts at the same level, s, each time the barrier S is crossed. Therefore, the inverse gaussian is in fact the density of all the waiting times between events. It is easy to verify that the counting process associated with the crossing times of this (S,s) rule is a Poisson process. Postulate 1 of the L.R.E. holds: The probability of two events occurring in the same time interval goes to zero as the size of the interval goes to zero whenever $s < S$. Postulate 2 holds because the process evolves without aftereffects: The probability of crossing the barrier S is independent of how long it took to previously cross it. Postulate 3 holds by construction: $\Pr[y^*(0) = S] = 0$ whenever $s < S$. Recall that from the definition of the Poisson process, Pr[S is not reached in the interval $[0,t)] = \Pr[N(t) = 0] = \exp\left\{-\int_0^t \lambda(u)du\right\}$. Therefore:

$$\exp\left\{-\int_0^t \lambda(u)du\right\} = 1 - \int_0^t \frac{(S-s)}{\sigma\sqrt{2\pi r^3}} \exp\left[\frac{((S-s)-\mu r)^2}{2\sigma^2 r}\right] dr \tag{28}$$

from where, in general $\lambda(t) = \lambda(t, s, S, \mu, \sigma)$. Each particular type of (S,s) rule produces inter-arrival times between decisions that are random variables. By inspecting whether the three postulates of the L.R.E. hold, one can justify that the associated counting process will be Poisson.

5 General Specification of Decision-Making

In the presence of random-time aggregation, we have determined that expressions of the form $E_{t-1}(y_t)$ are, in general, functions of past information, a parameter space, and the intensity of the decision process. This section considers more general intensity specifications that will allow us to examine scenarios in which: the frequency of decision making is time dependent (inhomogeneous Poisson processes); intensities that are a mixture of processes (mixed Poisson processes); and decision making that depends on an auxiliary information process such as stage of the business cycle effects, interest rates, unemployment effects, etc. (doubly-stochastic Poisson processes). In each case, the properties of the particular process will dictate the appropriate time series representation and the properties that we should expect our model to have.

The extensive literature on point processes (comprehensively surveyed in Snyder and Miller 1991) examines a wide variety of models and specifications. It usually considers the joint distribution of the Poisson process and its companion process (for example, the marks in a marked point process, the information process in a doubly stochastic process, etc.). Here, however, the focus is on the distribution of the Poisson Process, *conditional* on its companion process. In all instances, we will consider the case when this companion process is predetermined. By adopting this conditional approach, we simplify the problem tremendously and return the focus to the estimation of the partial adjustment model.

5.1 Temporal Poisson Processes

Consider the case where the intensity is a function of time. More generally:

$$\int_{t-1}^{t} \lambda(u)du = h(t) \tag{29}$$

This specification is appropriate in scenarios where, for example, there is a period of learning. At the start of the adjustment process, decisions are made more frequently. As the learning process evolves, the intensity converges to some average value. A typical specification that captures this effect is:

$$h(t) = \lambda + \gamma t^{\gamma - 1} \qquad \lambda > 0 \tag{30}$$

When $\gamma > 1$, the process is explosive, when $\gamma = 1$ yields a non-dependent process. For $\gamma < 1$, the process converges to λ as $t \to \infty$. Under this specification, estimation of the conditional expectation at time $t-1$ of our variable of interest can be done by Non-Linear Least Squares (NLLS) where the variance of the residuals becomes time-dependent.

Other popular specifications in the theory of point processes allow the intensity to depend on the number of past events (such as *renewal* processes, *birth-death* processes) and even their occurrence times (*self-exciting point processes*). However, in general, the sample path for the point process is not observed when data is sampled at regular intervals of time. Their usefulness is thus limited: these specifications involve unobserved latent processes that complicate matters considerably.

5.2 Mixed Poisson Processes

The possibility of asymmetric behavior is often considered in the literature of partial adjustment models. In the context of price adjustments in an environment with inflation, for example, being above some "optimal" price level is not as costly as being below it. In such cases, we should expect the frequency of positive corrections to differ from the frequency of negative corrections. Positive and negative disequilibria might not be the only two states that have different intensities. An appropriate specification for this type of problem is the *mixed Poisson process.* In general, the intensity of a mixed Poisson process can be expressed as:

$$\lambda(t) = \sum_{j=1}^{s} \lambda_j(t) \qquad s > 1 \tag{31}$$

The simplest case is to allow the intensity to have a constant value in each of the s states. The modelling stage can be approached from two angles: One is to allow λ_j to depend on predetermined variables. For example, if $z_{t-1} > 0$ choose $\lambda(t) = \lambda_1$; otherwise, choose $\lambda(t) = \lambda_2$. A more general approach however, is to let the transition between states follow a Markov hidden process. In this case, we can use the Switching-regimes model proposed by Hamilton (1989). An example with two states of Equation 7 would be:

$$\begin{aligned} z_t = \ & \mu_1 + \rho_1 z_{t-1} + \varepsilon_{1t} \qquad if \quad s_t = 1 \\ & \mu_2 + \rho_2 z_{t-1} + \varepsilon_{2t} \qquad if \quad s_t = 2 \end{aligned} \tag{32}$$

where we have allowed for two states with $\rho_i = \exp\{-\alpha\lambda_i\}, \quad i = 1, 2$. The variance of the error term will be also state dependent. The appeal of this approach is that we can obtain the identification of the transition probabilities directly from the estimation stage.

5.3 Doubly Stochastic Poisson Processes

An appealing specification of the P.A.M. is to allow the intensity of the decision-making process to depend on predetermined information. Decisions might be done more frequently in downturns than in expansions or when interest rates are low rather than high. Large disequilibria might be adjusted more frequently than small disequilibria. $\{N(t) : t \geq t-1\}$ is a doubly stochastic Poisson process with intensity process $\{\lambda(x(t), t) : t \geq t-1\}$ if for almost every given path of the process $\{x(t) : t \geq t-1\}$, $N(.)$ is a Poisson process with intensity function $\{\lambda(x(t),t) : t \geq t-1\}$. In other words, $\{N(t) : t \geq t-1\}$ is conditionally a Poisson process with intensity $\{\lambda(x(t),t) : t \geq t-1\}$ given $\{x(t) : t \geq t-1\}$. The process $\{x(t) : t \geq t-1\}$ is called the *information process.* In general:

$$\begin{aligned} \Pr[N(t) &= n] = E\left(\Pr[N(t) = n|x(u) : t-1 \leq u \leq t]\right) = \\ & E\left(\frac{1}{n!}\left[\int_{t-1}^{t} \lambda(x(u),u)du\right]^n \exp\left\{-\int_{t-1}^{t} \lambda(x(u),u)du\right\}\right) \end{aligned} \tag{33}$$

The expectation on the right hand side of Equation 33 is generally difficult, if not impossible, to evaluate. Therefore, we proceed by conditioning the Poisson process on the path of $x(t)$. The literature of duration data is rich in specifications (see Lancaster (1990)). The two most frequently used specifications are the proportional intensity model and the generalized accelerated failure time model.

1. **Proportional Intensity Model:** This parametrization allows separability of the effect of the predetermined covariates from the underlying decision process. The intensity is specified as:

$$\int_{t-1}^{t} \lambda(x(u),u)du = \lambda_0(t)k(x) \tag{34}$$

Usually $k(x) = \exp\{x'\delta\}$, allowing the parameter space of δ to be unconstrained. However, it is interesting to note that parametrizing $k(x) = \gamma(x-c)^2$ and $\lambda_0(t) = \lambda$, and specifying the model as in Equation 7 with zero constant (for simplicity), we have:

$$z_t = z_{t-1} \exp\left\{-\alpha\lambda\gamma(x_{t-d} - c)^2\right\} + \varepsilon_t \tag{35}$$

This is the specification of Exponential Smooth Autoregressive Models (ESTAR) – a popular family of non-linear time series models. Tests for non-linearity against this alternative, specification, estimation, and evaluation procedures are well developed for this type of model (See Teräsvirta 1994; Escribano and Jorda 1997). Under this specification, the errors present heteroskedasticity conditional on the information process.

2. **Generalized Accelerated Failure Time Model (G.A.F.T.):** an alternative specification to the proportional intensity model, the G.A.F.T. is suitable to model situations in which the effect of the explanatory variables acts multiplicatively on the time scale. The general specification of the G.A.F.T. is:

$$\int_{t-1}^{t} \lambda(x(u), u)du = \lambda_0(tk(x))k(x) \tag{36}$$

For many particular distributions (like the Exponential and the Weibull), the proportional intensity model and the G.A.F.T. produce similar specifications.

6 Detecting Random Time Aggregation

So far, we have shown that inference in the presence of random time aggregation can be severely biased and have given specific functional forms to these biases. However, is there a way to detect a priori random time aggregation? To answer this question we need to refer back to Equations 7 and 21, reproduced here for convenience:

$$z_t = (1-\alpha)^{k_t} z_{t-1} + \left\{ \mu \sum_{i=1}^{k_t} (1-\alpha)^i \eta_i + \mu \left[1 - \sum_{i=1}^{k_t} \eta_i \right] \right\} + \left\{ \sum_{i=1}^{k_t} (1-\alpha)^i \varepsilon_i + \varepsilon_{k_t+1} \right\}$$

$$\Delta y_t = \gamma(k_t) + \alpha^{k_t} \delta(k_t) z_{t-1} + \alpha^{k_t} \beta w_{t-1} + u_t$$

A typical scenario will contain z_t, y_t, and w_t as observable variables and k_t as an unobservable or latent process. When $k_t = k \ \forall t$ (the traditional type of time aggregation), the resulting equations are linear with constant variances and parameters, consequently virtually indistinguishable from a non-aggregated stochastic process. Unless the practitioner has ex-ante information regarding k, empirically there is not much that can be done. However, under the assumption that k is Poisson distributed, k_t will attain different values at each date t which means that the model will have time-varying parameters and the error terms will most certainly be heteroskedastic. These two clues are useful in suspecting random-time aggregation. Unfortunately, for a highly aggregated series there will be little variation of the k_t over time (by the properties of the Poisson) and the variance of the error terms will be relatively large, thus reducing the signal to noise ratio.

The modeling strategy proposed here is to proceed conditionally on k_t and use any of the inhomogeneous intensity specifications described in the previous section. The resulting time-series models are typically non-linear, a feature for which there generally exists a testing methodology. This is the strategy that will be used the following application to a model of inventories.

7 Empirical Application: Inventory Adjustment

Inventory research to this date has centered mainly in two microeconomic models of firm behavior: the production smoothing model and the (S,s). The standard empirical specification since Lovell (1961) has been the stock adjustment equation briefly introduced in Section 3.2, which we reproduce here with slightly different notation:

$$I_t - I_{t-1} = \phi_0 + \phi_1 \left(I^*_{t-1} - I_{t-1}\right) - \phi_2 \left(S_{t-1} - S^e_{t-1}\right) + u_t \tag{37}$$

where I_t = stock of inventories at time t; I^*_t = desired level of inventories at time t; S_t = sales and S^e_t = expected sales. While the predictions of this model are simple and intuitive, three major findings have clouded its success. First, adjustment speeds (given by ϕ_1) are extremely low. While even the wildest swings in inventory stocks amount to only a few days production, adjustment speeds are usually estimated to be less than 10% a month. Efforts to explain this finding as a result of some econometric bias have had limited success. Economically, slow adjustment speeds could be explained by firms' desire to smooth production. However, two additional empirical findings also contradict this view. Production is found to be more volatile than sales and unanticipated sales shocks (i.e. $S_{t-1} - S^e_{t-1}$) do not seem to lead to inventory disinvestment.

In this section, a stock adjustment model similar to 37 is estimated. In particular, we analyze data on glass containers as provided by the Census Bureau's monthly survey: Manufacturers' Shipments, Inventories and Orders, otherwise known as the M-3 report. The data span from 1991 to 1996 at a monthly frequency. There are several advantages to this data set. First, it is obtained from a monthly survey of 16 known manufacturers, thereby reducing problems of aggregation across firms with different cost structures. Second, the data is in physical units (more specifically in thousands of gross, one thousand gross = 144,000) which is helpful in view of Krane and Braun (1991) who report that different adjustments and accounting methods to value-based data typically introduce measurement error. Third, the data is not seasonally adjusted and therefore exempt of the distortions typical of conventional seasonal adjustment methods.

We begin by reporting the basic properties of the data. Figure 3 displays the series of production, shipments and inventories. The series exhibit high seasonal fluctuations but appear otherwise stationary (an appreciation further confirmed by augmented Dickey-Fuller tests and Phillips-Perron tests that overwhelmingly reject the null hypothesis of a unit root). Inventory dis/investment is relatively small compared to production (the largest of deviations approximately correspond to two days production). The ratio of the variance of production to sales is 0.90 which supports the notion that inventories are partly used to buffer sales fluctuations. Interestingly, if the data is deseasonalized (additively with a set of seasonal dummies), this ratio becomes 1.25 a frequent finding in the literature on inventories. Further support for the buffer role of inventories is provided in Figure 4. The top graph displays the seasonal means of Production and Shipments while the bottom graph displays their difference (i.e., inventory investment due to seasonal fluctuations). Seasonal fluctuations account for approximately 40% of the variation in inventory investment. Adjusting for the number of days in each month, production remains approximately constant from February to November and drops in December and January. On the other hand, shipments drop from November to

February and are relatively higher during the summer months.

With this is mind, consider estimation of Equation 37. Traditionally, it is assumed that the desired level of inventories can be approximated by a linear function of sales. In addition, sales expectations are also constructed on the basis of past observations on sales. Without imposing any restriction, 37 can then be rewritten as,

$$\begin{aligned} \Delta I_t &= c_0 - \phi_1 I_{t-1} + \psi_1 S_{t-1} + ... + \psi_p S_{t-1} + \varepsilon_t \\ \varepsilon_t &= \rho \varepsilon_{t-1} + u_t \end{aligned} \tag{38}$$

where the parameter of interest is ϕ_1, the speed of adjustment parameter. While this formulation is fully flexible, one cannot identify the "buffer" parameter unless coefficient restrictions are imposed. The dynamic structure of the error term stems from the fact that I_t^* is being approximated. In addition, u_t is allowed to contain the seasonal component of inventory investment described above. Table 1 reports the estimation results of the relevant parameters.

The estimate $\widehat{\phi}_1 = 0.21$ attains a relatively high value relative to what is common in this literature yet still low to be consistent with the theory and our preliminary findings. This point is best illustrated by calculating the number of days required to close 95% of a given disequilibrium. This quantity can be calculated as $T = 30log(0.05)/log(1 - \widehat{\phi}_1)$ which for this example becomes 381 days. The seasonal pattern of u_t matches that reported in Figure 4. The seasonal variation is reduced to 76% of the original

Next we want to check whether random-time aggregation might be driving these results. We propose a conditional, proportional hazard model similar to that proposed in Section 5.3:

$$k_t = exp\left\{-\gamma(\Delta I_{t-d} - \delta)^2\right\}$$

so that the stock adjustment model 37 becomes:

$$\begin{aligned} \Delta I_t &= exp\left\{-\gamma(\Delta I_{t-d} - \delta)^2\right\} \{c_0 - \phi_1 I_{t-1} + \psi_1 S_{t-1} + ... + \psi_p S_{t-1}\} + \varepsilon_t \qquad (39) \\ \varepsilon_t &= \rho\varepsilon_{t-1} + u_t \end{aligned}$$

which is a special case of smooth transition autoregressive model (introduced by Teräsvirta 1994) for which Escribano and Jordá (1997) have developed the specification tests that we use here. First we test the series of changes in inventories for evidence of this type of nonlinearity. The auxiliary regression then becomes,

$$\Delta I_t = \sum_{i=1}^{12} \beta_i d_{it} + \omega_1 \Delta I_{t-1} + ... + \omega_4 \Delta I_{t-4} + \pi_1 \Delta I_{t-d}^2 + \pi_2 \Delta I_{t-d}^3 + \pi_3 \Delta I_{t-d}^4 + \nu_t \qquad (40)$$

where the seasonal dummies (d_i for $i = 1, 2, ..., 12$) are included to avoid spurious detection of nonlinearities. The non-linearity test simply consists of jointly testing the significance of the coefficients π_i ($i = 1, 2, 3$) and choosing d that minimizes the p-value of this test for $d = 1, 2, 3, 4$. A likelihood ratio test showed evidence of nonlinearity at the conventional 5% confidence level for $d = 3$.

Accordingly, Equation 39 was estimated by Weighted Non-linear Least Squares and White's covariance consistent estimates for the standard errors. (ΔI_{t-3} is divided by its standard deviation for computational convenience). Table 2 reports the result of this estimation. Unfortunately, the likelihood ratio test for the joint significance of the extra two regressors has a non-standard distribution (this problem is discussed in Escribano and Jordá 1997) but the T-statistics on the additional parameters are highly significant. The residuals exhibit the same seasonal pattern as before but the seasonal variation is further reduced to 72% of the original. The most revealing feature is the estimate $\widehat{\phi}_1 exp\left\{-\widehat{\gamma}(\Delta I_{t-d}-\widehat{\delta})^2\right\}$, the time-varying speed of adjustment displayed in Figure 5. The associated speed of adjustment fluctuates quite remarkably often obtaining values circa 0.70, a number much more in line with the predictions of the stock-adjustment model. The average adjustment for the period is around 0.18, fairly close to the estimated 0.21 with the linear specification. While by no means an exhaustive analysis, this relatively simple approach illustrates how random-time aggregation impacts inference on adjustment speeds. Alternatively, one could view our application as resulting from a misspecification of the theoretical model in that the adjustment speed is not allowed to depend on previous inventory investment..

8 Conclusion

This paper identified a new source of misspecification in econometric analysis: random-time aggregation. This misspecification arises when we abandon the assumption that the point-in-time sample intervals coincide *precisely* with the timing of agents' decisions. The bias arises from ignoring that economic decision making is done randomly in time while data collection is done at regular intervals of time. This common type of misspecification was analyzed in the context of partial adjustment models but is likely to affect many other situations in economics.

Partial adjustment models, popular reduced-form models of economic behavior, provide formal micro-foundations of decision processes that evolve in random-time. This paper demonstrated that the decision-making process is best understood as a point process. In particular, it was shown that (S,s) rules comply with the conditions of The Law of Rare Events, a theorem that states the conditions under which the Poisson appropriately represents the evolution of adjustments through time.

With the decision process explicitly modelled, estimation of the partial adjustment model involves joint modelling of the distribution of adjustments and decisions. However, typical empirical analysis disregards the latter. Consequently, speeds of adjustment are biased – the misspecification arises because the time $t-1$ expectation, unconditional with respect to the decision process, becomes a function of the intensity of this decision process and the proportion of adjustment. Correlations and/or causal relations are also incorrectly evaluated – as the intensity of decision making grows, so does the downward bias introduced, which in turn causes false rejections of these relations – due to the interaction between adjustments and aggregation. Random-time aggregation is not an artifact of low/high frequency sampling: it is an artifact of fixed interval sampling.

The models proposed are generalized to allow for varying decision-making intensities in three ways: First, time-dependent Poisson processes (useful to model learning effects) were proposed; second, mixed Poisson processes were considered – Markov switching models provided the appropriate technology for this type of generalization; and third, doubly stochastic Poisson processes are analyzed – while several specifications are available in the literature of duration data, smooth transition regression models provide sufficient flexibility to model this type of behavior.

An analysis of inventory behavior in the Glass Containers industry illustrates the practical implications of the previous claims. A simple test for non-linearity indicates possible misspecification due to random-time aggregation. Using lagged inventory investment as an explanatory covariate for variable speeds of adjustment, the stock-adjustment model is shown to perform as predicted by the theory.

Further areas of research span in two major directions – the effect of cross-sectional aggregation in the presence of random-time aggregation and the analysis of random-time aggregation in general time series models. The analysis in this paper focused on the effects of random-time aggregation at the level of the individual agent which is not independent of aggregation across heterogeneous agents. The message of this paper is clear: empirical analysis of economic behavior proceeds by adapting the economic model to the available data. Consistently ignoring the way that data is generated results in severe biases in structural inference of econometric models. This paper provides an example of how such misspecifications arise when we ignore the time dimension of the problem.

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9 Appendix

Detailed derivation of Equations 10 and 11 follow. From Equation 9:

$$\begin{aligned} E_{t-1}(z_t) &= P_0\left[z_{t-1}+\mu+\sigma\int_{t-1}^{t}d\xi(r)\right]+ \\ &\quad P_1\left[(1-\alpha)\left\{z_{t-1}+\mu\eta_1+\sigma\int_{t-1}^{\tau_1}d\xi(r)\right\}+\mu(1-\eta_1)+\sigma\int_{\tau_1}^{\tau_2}d\xi(r)\right]+... \end{aligned} \tag{41}$$

Therefore:

$$E_{t-1}(z_t)=\exp\left\{-\int_{t-1}^{t}\lambda(r)dr\right\}z_{t-1}\left[\begin{array}{c} 1+(1-\alpha)\int_{t-1}^{t}\lambda(r)dr+ \\ \frac{(1-\alpha)^2}{2!}\left(\int_{t-1}^{t}\lambda(r)dr\right)^2+... \end{array}\right]+\overline{\mu}+E_{t-1}(\varepsilon_t) \tag{42}$$

where

$$\begin{aligned} \overline{\mu} &= \mu\exp\left\{-\int_{t-1}^{t}\lambda(r)dr\right\} \\ &\left[\begin{array}{c} 1+\left((1-\alpha)\overline{\eta}_1+(1-\overline{\eta}_1)\int_{t-1}^{t}\lambda(r)dr\right)+ \\ ((1-\alpha)^2\overline{\eta}_1+(1-\alpha)\overline{\eta}_2+(1-\overline{\eta}_1-\overline{\eta}_2))\frac{\left(\int_{t-1}^{t}\lambda(r)dr\right)^2}{2!}+... \end{array}\right] \end{aligned} \tag{43}$$

with $E_{t-1}(\eta_i)=\overline{\eta}_i$; and

$$\begin{aligned} \varepsilon_t &= \sigma\exp\left\{-\int_{t-1}^{t}\lambda(r)dr\right\} \\ &\left[\int_{t-1}^{t}d\xi(r)+\left\{(1-\alpha)\int_{t-1}^{\tau_1}d\xi(r)+\int_{\tau 1}^{t}d\xi(r)\right\}\int_{t-1}^{t}\lambda(r)dr+...\right] \end{aligned} \tag{44}$$

Now, it is easy to see that:

$$
\begin{aligned}
E_{t-1}(z_t) &= \exp\left\{-\int_{t-1}^{t}\lambda(r)dr\right\} z_{t-1}\exp\left\{(1-\alpha)\int_{t-1}^{t}\lambda(r)dr\right\} \\
+\overline{\mu}+E_{t-1}(\varepsilon_t) &= z_{t-1}\exp\left\{-\alpha\int_{t-1}^{t}\lambda(r)dr\right\}+\overline{\mu}+E_{t-1}(\varepsilon_t)\ \blacksquare
\end{aligned} \tag{45}
$$

and Equation 11 immediately follows from 43. Note that since $E_{t-1}\left(\int_{\tau_{i-1}}^{\tau_i} u(r)dr\right) = 0\ \forall i = 0,1,2,...$ then $E_{t-1}(\varepsilon_t) = 0$, where:

$$
\varepsilon_t = \sigma \exp\left\{-\alpha\int_{t-1}^{t}\lambda(r)dr\right\}\left(\sum_{i=0}^{\infty}\frac{\left(\int_{t-1}^{t}\lambda(r)dr\right)}{i!}\left[\sum_{j=0}^{i}(1-\alpha)^j\int_{\tau_{i-j}}^{\tau_{i+1-j}}d\xi(r)\right]\right)
$$

We state a theorem to calculate $\overline{\eta}_i$.

Theorem 4 *: Let $\eta_1, \eta_2, ...,$ be the occurrence times in a Poisson process of rate $\dot{\lambda} > 0$. Conditioned on $N(t) = n$, the random variables $\eta_1, ..., \eta_n$ have joint probability density function:*

$$
f_{\eta_1,...,\eta_n|N(t)=n}(\eta_1,...,\eta_n) = n!t^{-n} \qquad for \quad 0 < \eta_1 < ... < \eta_n < t \tag{46}
$$

that is, they are jointly, uniformly distributed.

Proof: *Taylor and Karlin (1994) p.270*

References

Abraham, Katherine G. and Susan S. Housman, 1993, Does Employment Protection Inhibit Labor Market Flexibility? Lessons from Germany, France and Belgium. *Working Paper No. 4390.* NBER, Cambridge, MA.

Andersen, P.K., Borgan, ., Gill, R.D. and Keidig, N. (1992), *Statistical Models based on Counting Processes*, Springer Series in Statistics, New York: Springer-Verlag.

Bergstrom, A.R. (1984), "Continuous-Time Stochastic Models and Issues of Aggregation over Time," in *Handbook of Econometrics*, Ed.. Z. Griliches and M.D. Intrilligator, New York: Elsevier Science Publishers, North-Holland.

Blinder, Alan S, 1981, Retail Inventory Behavior and Business Fluctuations. *Brookings Papers on Economic Activity*, Vol. 2:1981, 443-505.

Blinder, Alan S., 1986, Can the Production Smoothing Model of Inventory Behavior Be Saved? *Quarterly Journal of Economics*, Vol. 101, August, 431-453.

Blinder, Alan S. and Louis J. Maccini, 1991, Taking Stock: A Critical Assessment of Recent Research on Inventories. *Journal of Economic Perspectives*, Vol. 5, No. 1 - Winter 1991, 73-96.

Caballero, Ricardo J. and Eduardo M.R.A. Engel, 1993, Microeconomic Adjustment Hazards and Aggregate Dynamics. *Quarterly Journal of Economics*, Vol. 108, No. 2, May, 313-358.

Caballero, Ricardo J. and Eduardo M.R.A. Engel, 1994, Explaining Investment Dynamics in U.S. Manufacturing: A Generalized (S,s) Approach. Working Paper.

Caballero, Ricardo J., Eduardo M.R.A. Engel and John Haltiwanger, 1994, Aggregate Employment Dynamics: Building from Microeconomic Evidence. *Economic Fluctuations Research Meeting, NBER*, July 15-16.

Christiano, Lawrence J. and Martin Eichenbaum, 1987, Temporal Aggregation and Structural Inference in Macroeconomics. *Carnegie-Rochester Conference Series on Public Policy*, Vol. 26, 63-130.

Escribano, Alvaro and Oscar Jorda, 1997, Improved Testing and Specification of Smooth Transition Regression Models, *University of California, Davis*, Working Paper 97-26.

Feldstein, Martin and Alan Auerbach, 1976, Inventory Behavior in Durable Goods Manufacturing: The Target Adjustment Model. *Brookings Papers of Economic Activity,* Vol. 2:1976, 351-396.

Hamermesh, Daniel S., 1989, Labor Demand and the Structure of Adjustment Costs. *American Economic Review*, Vol. 79, No.4, September, 674-689.

Hamermesh, Daniel S., 1993, Labor Demand and the Source of Adjustment Costs. *Economic Journal*, 105(430), May, 620-34.

Hamilton, James D., 1989, A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle. *Econometrica*, 57 (2), March, 357-84.

Karlin, Samuel and Howard M. Taylor, 1975, *A First Course in Stochastic Processes.* (Academic Press Ed.).

Karlin, Samuel and Howard M. Taylor, 1981, *A Second Course in Stochastic Processes.* (Academic Press Ed.).

Krane, Spencer D. and Stephen N. Braun, 1991, Production Smoothing Evidence from Physical-Product Data. *Journal of Political Economy,* 99(3), June, 558-81.

Lancaster, Tony, 1990, *The Econometric Analysis of Transition Data.* Econometric Society Monographs, (Cambridge Ed.).

Lovell, Michael C., 1961, Manufacturers' Inventories, Sales Expectations and the Acceleration Principle. *Econometrica,* XXIX, July, 293-314.

Meese, Richard, 1980, Dynamic Factor Demand Schedules for Labor and Capital under Rational Expectations. Journal *of Econometrics* Vol. 14:141-58.

Nickell, S. J., 1984, An Investigation of the Determinants of Manufacturing Employment in the United Kingdom. *Review of Economic Studies* 51, 529-57.

Nickell, S. J., 1986, Chapter 9, Dynamic Models of Labor Demand, in: O. Ashenfelter and R. Layard eds., *The Handbook of Labor Economics*, Vol. I, .(Elsevier Science Publishers), 474-522.

Phipps, Anthony J., 1975, The Relationship between Output and Employment in British Manufacturing Industry. *Oxford Bulletin of Economics and Statistics,* 37(1), February, 49-63.

Rotemberg, Julio J., 1987, The New Keynesian Microfoundations, in: O.J. Blanchard and S. Fischer eds., *NBER Macroeconomics Annual,* 1987, 69-116.

Sargent, Thomas J., 1978, Estimation of Dynamic Labor Demand Schedules Under Rational Expectations. *Journal of Political Economy,* 86, 1009-44.

Sims, Christopher A., 1974, Output and Labor Input in Manufacturing. *Brookings Papers on Economic Activity*: 3, 695-728.

Smyth, David J., 1986, The Cyclical Response of Employment to Output Changes: U.S. Manufacturing Industries 1948 to 1983. *Applied Economics*, May, Vol. 18, No. 5, 495-500.

Snyder, Donald L. and Michael I. Miller, 1991, *Random Point Processes in Time and Space*. Second Edition, New York, (Springer Verlag).

Stock, James H., 1987, Measuring Business Cycle Time. *Journal of Political Economy*, Vol. 95, No. 6, 1240-1261.

Stock, James H., 1988, Estimating Continuous-Time Processes Subject to Time Deformation. An Application to Postwar U.S. GNP. *Journal of the American Statistical Association*, Vol. 83, No. 401, March, 77-84.

Taylor, Howard M. and Samuel Karlin, 1994, *An Introduction to Stochastic Modelling*, (Academic Press).

Teräsvirta, T., 1994, Specification, Estimation and Testing of Smooth Transition Autoregressive Models. *Journal of the American Statistical Association*, 89(425), March, 208-218.

Figure 1: Shape of the Bias in the Speed of Adjustment Parameter Under Random Time Aggregation (Autoregressive Specification).

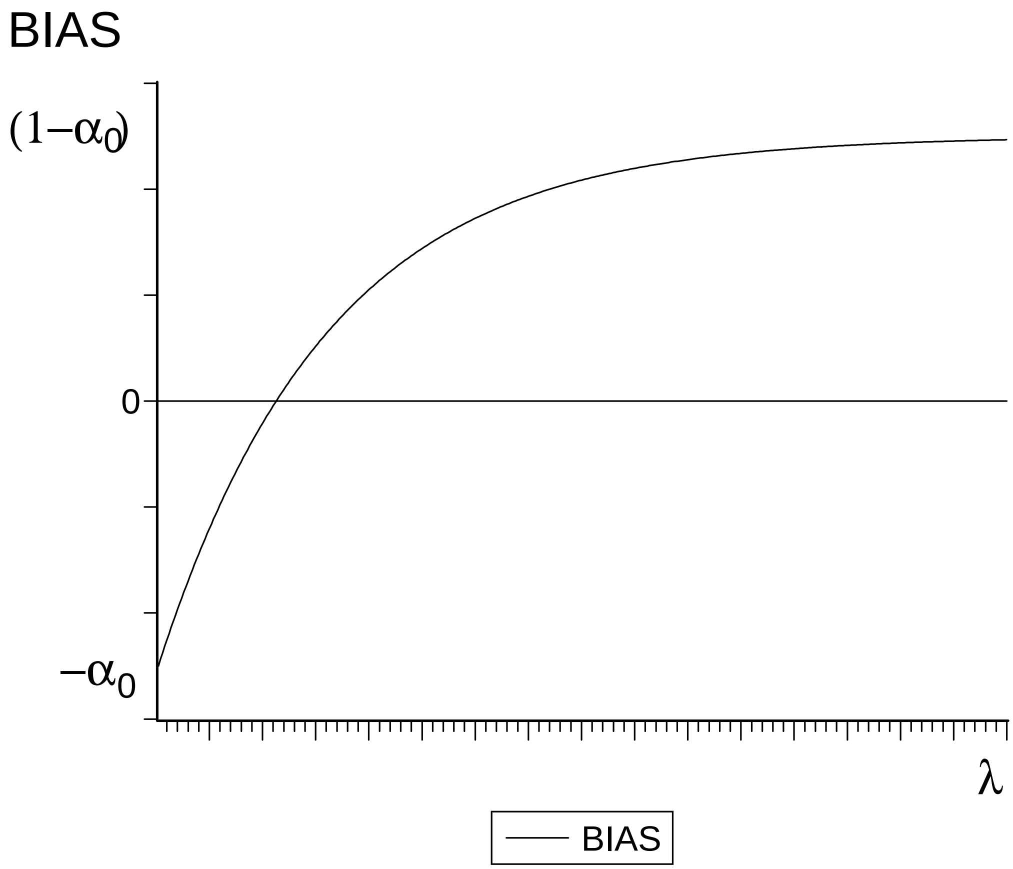
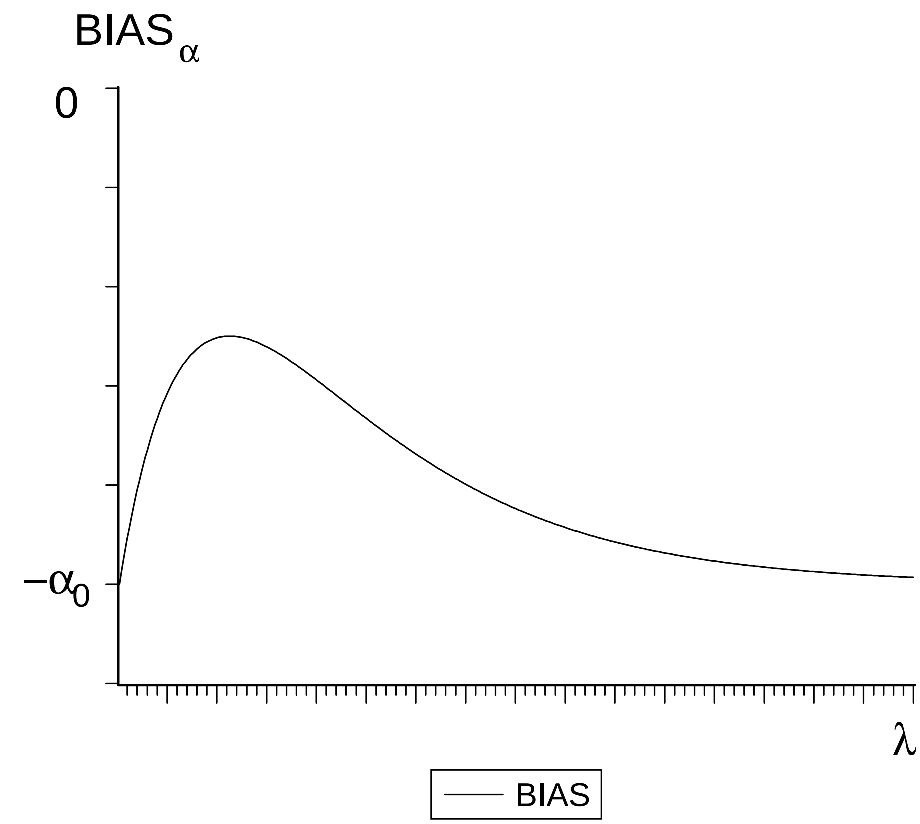


Figure 2: Shape of the Bias in the Alpha and Beta Parameters Under Random Time Aggregation (Specification includes Exogenous Variable)



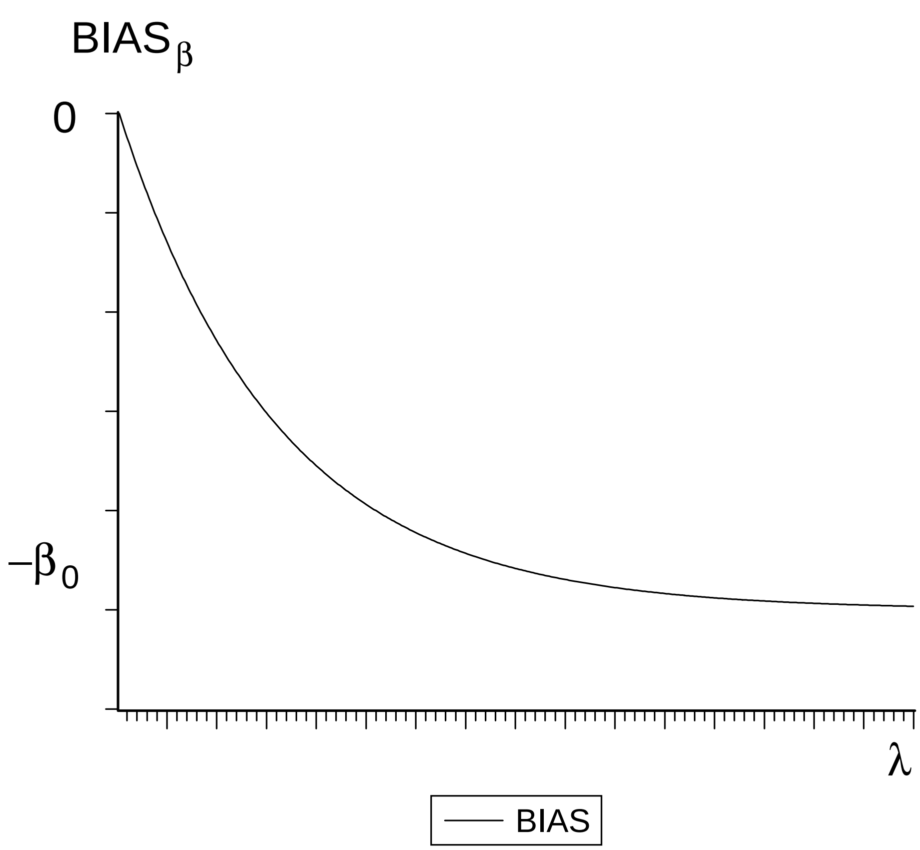


Figure 3: Production, Shipments and Inventories of Glass Containers. SAMPLE: 1991:1 to 1996:12. NSA

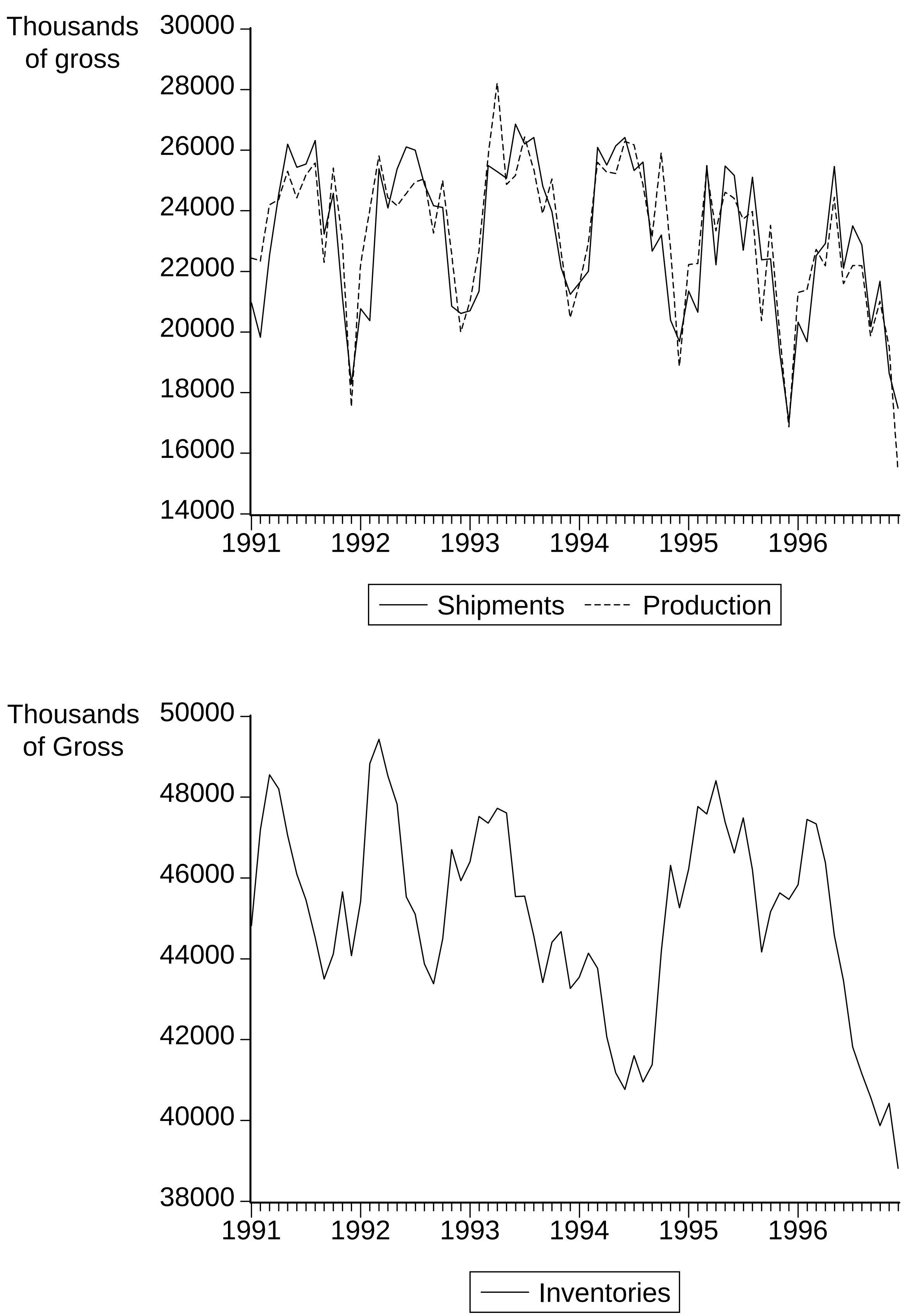
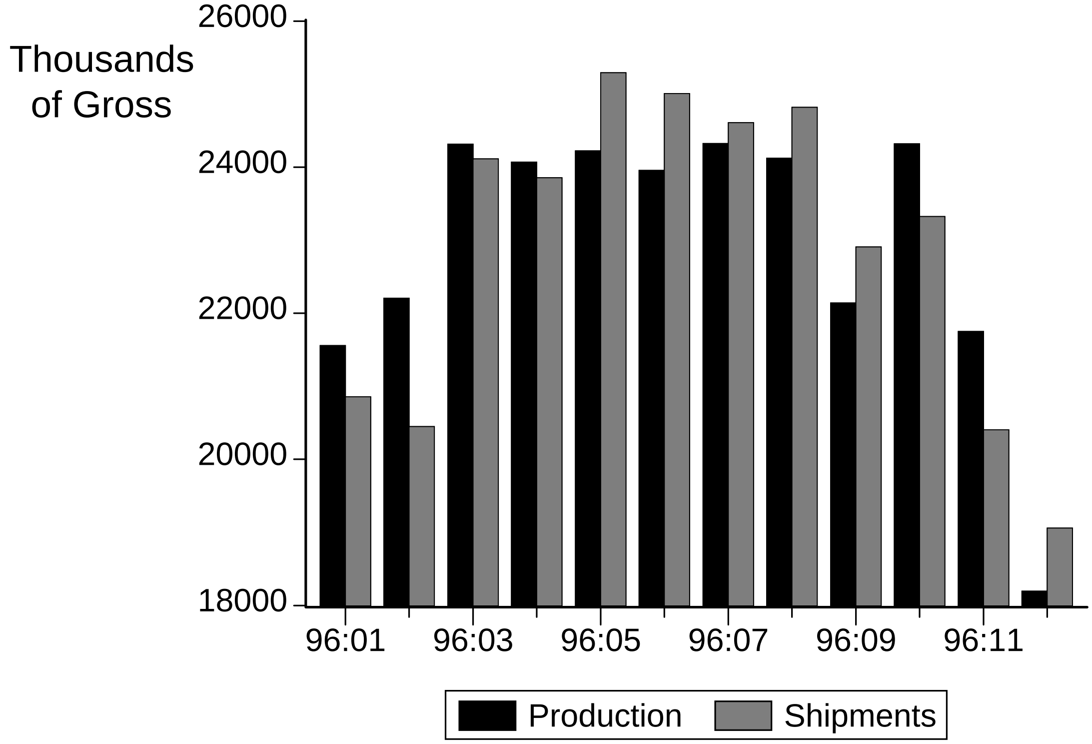


Figure 4: Seasonal Means of Production, Shipments and Inventory Investment of Glass Containers. Sample: 1991:1 to 1996:12, NSA.



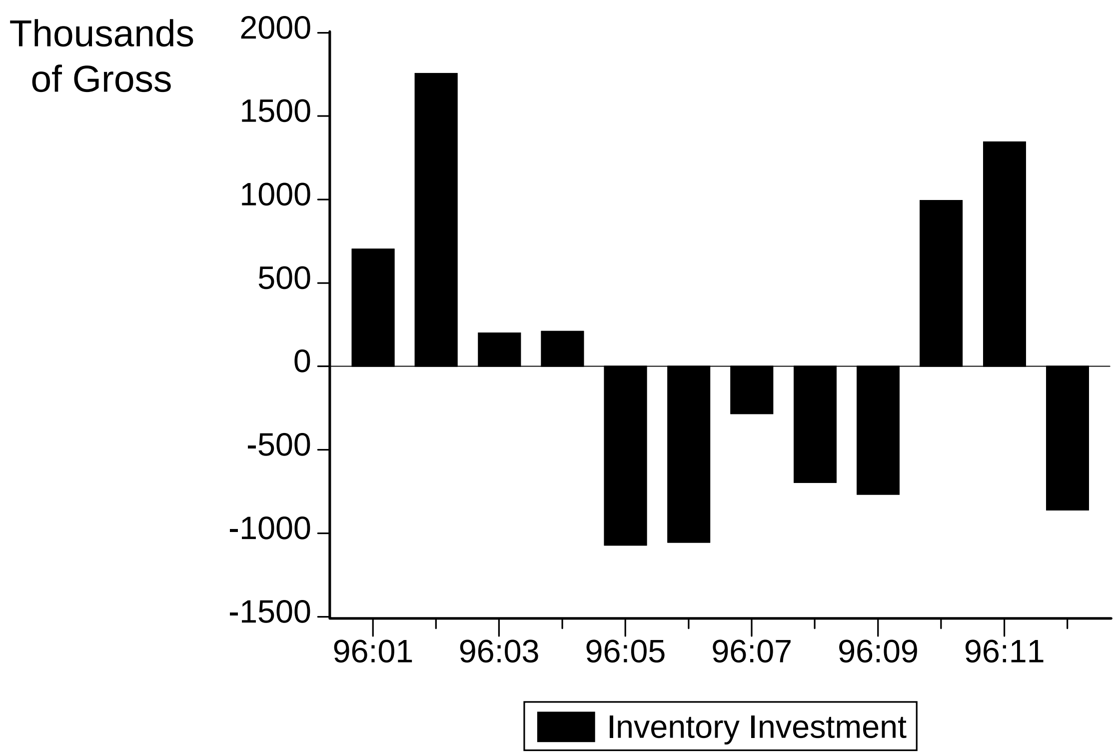


Figure 5: Estimated Adjustment Parameter from the Nonlinear Specification of the Stock Adjustment Model

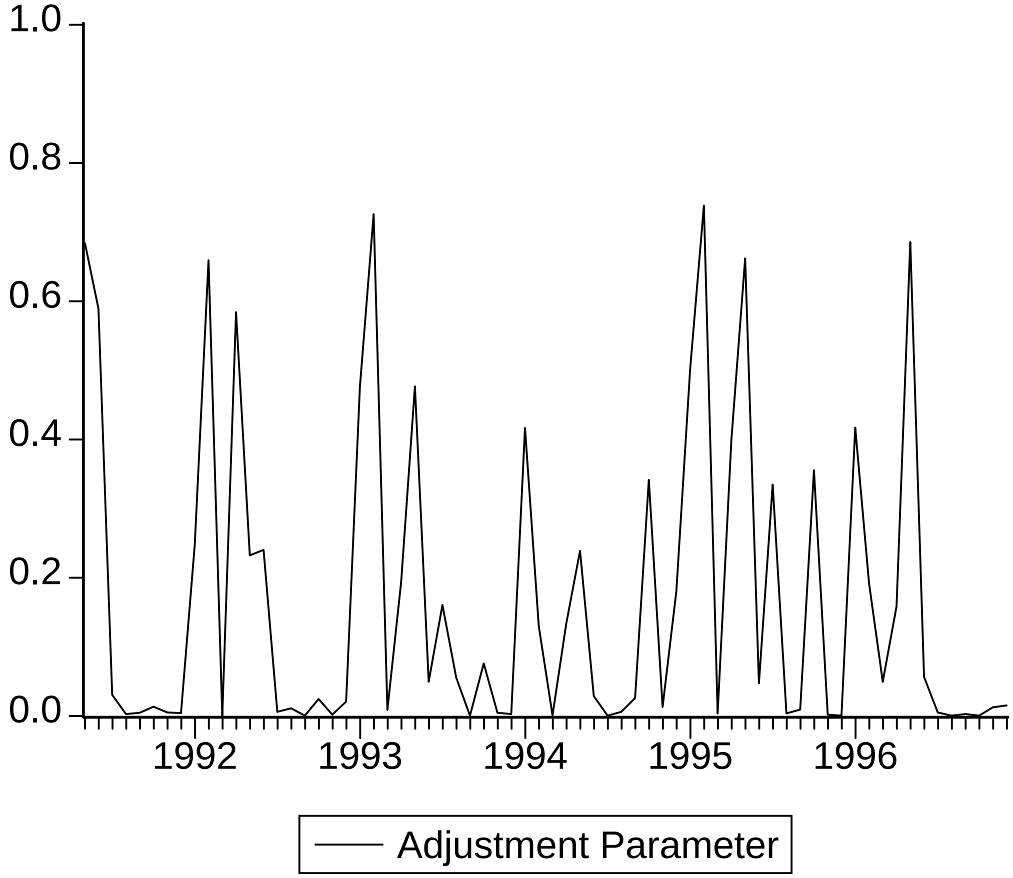


Table 1: Estimates of the Linear Stock Adjustment Model for Inventories of Glass Containers. Sample: 1991:1 to 1996:12, N.S.A.

Coefficient	**Estimate (Std. Error)**	**T-Statistic**
$\mathbf{C_0}$	13427.17	3.0477
	(4405.667)	
$\mathbf{f_1}$	0.2085	2.5006
	(0.0834)	
$\mathbf{y_1}$	-0.0007	0.0102
	(0.0636)	
$\mathbf{y_2}$	-0.1781	2.7152
	(0.0656)	
r	0.4634	3.7549
	(0.1234)	
January	741.25	1.9995
	(370.72)	
February	830.23	2.2394
	(370.72)	
March	-40.50	0.1197
	(338.42)	
April	-300.08	0.8867
	(338.42)	
May	-69.77	0.2062
	(338.42)	
June	-780.79	2.3071
	(338.42)	
July	659.79	1.9496
	(338.42)	
August	-884.28	2.6129
	(338.42)	
September	-302.99	0.8953
	(338.42)	
October	1098.38	3.2456
	(338.42)	
November	458.29	1.3542
	(338.42)	
December	-1209.34	3.5735
	(338.42)	
Log-Likelihood	-563.1564	
SSR	39856715	
Durbin-Watson	1.8011	
R^2	0.60	

Standard Errors in parenthesis

Table 2: Estimates of the Non-Linear Stock Adjustment Model for Inventories of Glass Containers. Sample: 1991:1 to 1996:12.

Coefficient	Estimate	T-Statistic
C_0	41369.68	3.6205
	(11426.43)	
f_1	0.7444	2.7340
	(0.2723)	
y_1	-0.0222	0.1500
	(0.1481)	
y_2	-0.2659	2.0650
	(0.1288)	
r	0.2726	2.3377
	(0.1166)	
g	0.8722	1.8453
	(0.4727)	
d	1.6294	14.8792
	(0.1095)	
January	-147.71	0.4340
	(340.37)	
February	437.57	1.2856
	(340.37)	
March	-501.21	1.4725
	(340.37)	
April	-321.36	0.9442
	(340.37)	
May	-301.12	0.9691
	(310.72)	
June	-866.62	2.7891
	(310.72)	
July	224.63	0.7229
	(310.72)	
August	-913.06	2.9386
	(310.72)	
September	-572.70	1.8431
	(310.72)	
October	801.43	2.5793
	(310.72)	
November	897.62	2.8889
	(310.72)	
December	-1447.97	4.6601
	(310.72)	
Log-Likelihood	-541.0503	
SSR	32439119	
Durbin-Watson	1.9436	
R^2	0.67	

Standard Errors in parenthesis