# Problem Set 3 Solution

**1.**

**(a)** The problem is

$$\min Q(\theta) = g(\widehat{\pi}_T;\theta)'\widehat{W}g(\widehat{\pi}_T;\theta).$$

We will use some auxillary assumptions,

$$\widehat{W} \longrightarrow W \text{ and } \sqrt{T}(\widehat{\pi}_T - \pi_0) \xrightarrow{d} N(0,\sigma^2),$$

along with the usual regularity conditions.

The FOC is

$$-\widehat{G}'\widehat{W}g = 0$$

where $\widehat{G} = dh(\theta)/d\theta$. Note that we have a CLT result for $\widehat{\pi}_T - h(\theta_0)$ but not for $\widehat{\pi}_T - h(\widehat{\theta})$. Hence, apply the mean value theorem, we get

$$\widehat{\pi}_T - h(\widehat{\theta}) = [\widehat{\pi}_T - h(\theta_0)] - \frac{dh(\bar{\theta})}{d\theta}(\widehat{\theta}_T - \theta_0).$$

Plugging this expression back into the FOC leads to

$$-\frac{dh(\widehat{\theta})}{d\theta}'\widehat{W}\left\{[\widehat{\pi}_T - h(\theta_0)] - \frac{dh(\bar{\theta})}{d\theta}(\widehat{\theta}_T - \theta_0)\right\} = 0.$$

If $\widehat{\theta}_T \longrightarrow \theta_0$ and $h(\cdot)$ is continuously differentiable, then

$$\frac{dh(\widehat{\theta})}{d\theta}' \text{ and } \frac{dh(\bar{\theta})}{d\theta}' \xrightarrow{p} \frac{dh(\theta_0)}{d\theta}' = G'.$$

Hence

$$\sqrt{T}(\widehat{\theta}_T - \theta_0) = -\sqrt{T}(\widehat{G}'\widehat{W}\widehat{G})^{-1}\widehat{G}'\widehat{W}[\widehat{\pi}_T - h(\theta_0)]$$

and

$$\sqrt{T}(\widehat{\theta}_T - \theta_0) \xrightarrow{d} N\left(0, (G'WG)^{-1}G'W\sigma^2WG(G'WG)^{-1}\right).$$

When $W = (\sigma^2)^{-1}$, this reduces to

$$\sqrt{T}(\widehat{\theta}_T - \theta_0) \xrightarrow{d} N\left(0, (G'(\sigma^2)^{-1}G)^{-1}\right).$$

**(b)** Now $h(\theta) = 3\theta^2$ and $dh(\theta)/d\theta = 6\theta = G$. So $\sqrt{T}(\widehat{\theta}_T - \theta_0) \xrightarrow{d} N\left(0, \sigma^2(36\theta_0^2)^{-1}\right)$.

**(c)** To use delta method, we need to express $\theta$ as a function of $\pi$.

$$\begin{aligned} \widehat{\pi}_T &= 3\theta^2 \Longrightarrow \theta = f(\pi) = \left(\frac{1}{3}\widehat{\pi}_T\right)^{1/2}. \\ f_\pi &= \frac{1}{6}\left(\frac{1}{3}\widehat{\pi}_T\right)^{-1/2} \xrightarrow{p} \frac{1}{6}\theta_0^{-1}. \end{aligned}$$

Thus $\sqrt{T}(\widehat{\theta}_T - \theta_0) \xrightarrow{d} N(0, f_\pi' Var(\widehat{\pi}) f_\pi) = N\left(0, \sigma^2(36\theta_0^2)^{-1}\right)$, which is the same as the answer in (b).

**2.** See Hamilton's book p.127-129 for details.

**3.**

**(a)** From assumptions in the problem,

$$\widehat{u}_t = y_t - \widehat{\rho}_1 y_{t-1} - \dots - \widehat{\rho}_p y_{t-p}.$$

$y_t$ is MA(1) with $|\theta| < 1$, so $y_t/(1+\theta L) = \epsilon_t$. With the condition $\widehat{\rho}_j \xrightarrow{p} (-1)^j\theta^j$, we can obtain

$$y_t = \theta y_{t-1} + \dots + (-1)^p\theta^p y_{t-p} + \sum_{j=p+1}^{\infty} (-1)^j\theta^j y_{t-j} + \epsilon_t.$$

Hence

$$\widehat{u}_t \xrightarrow{p} \sum_{j=p+1}^{\infty} (-1)^j\theta^j y_{t-j} + \epsilon_t.$$

Note that

$$\sum_{j=p+1}^{\infty} (-1)^j\theta^j y_{t-j} = \theta^{p+1}\sum_{j=0}^{\infty}(-1)^j\theta^j y_{t-p-j-1} = \theta^{p+1}(\epsilon_{t-p} + \theta\epsilon_{t-p+1}).$$

Hence

$$\widehat{u}_t \xrightarrow{p} \theta^{p+1}(\epsilon_{t-p} + \theta\epsilon_{t-p+1}) + \epsilon_t = w_{t-p} + \epsilon_t.$$

Since $|\theta| < 1$, we know that $w_{t-p} \xrightarrow{p} 0$ if $p \longrightarrow \infty$.

Remarks:

(1) Here the asymptotics can be deceiving. On the one hand, the quality of the first step approximation improves as $p \longrightarrow \infty$. However, in small samples, standard errors for $\widehat{\rho}_j$ grow as $p \longrightarrow \infty$, and hence the standard errors for $\widehat{u}_t$.

(2) Asymptotically, and with quite a bit of work, it can be shown that the right rate for $p \longrightarrow \infty$ is $p^2/T \longrightarrow 0$.

**(b)** The second step regression is $y_t = \theta \widehat{u}_{t-1} + v_t$. Thus

$$\begin{aligned}\widehat{\theta} &= \left(\frac{1}{T}\sum_1^T \widehat{u}_{t-1}^2\right)^{-1}\left(\frac{1}{T}\sum_1^T \widehat{u}_{t-1} y_t\right)\\ &= \left(\frac{1}{T}\sum_1^T \widehat{u}_{t-1}^2\right)^{-1}\left(\frac{1}{T}\sum_1^T \widehat{u}_{t-1}(\epsilon_t + \theta\epsilon_{t-1})\right).\end{aligned}$$

Note that

$$\frac{1}{T}\sum_1^T \widehat{u}_{t-1}^2 \xrightarrow{p} E\left\{\left[\epsilon_{t-1} + \theta^{p+1}(\epsilon_{t-p-1} + \theta\epsilon_{t-p-2})\right]^2\right\} = \sigma^2[1 + \theta^{2p+2}(1+\theta^2)] < \infty.$$

When $p \longrightarrow \infty$, the higher order term dies out. On the other hand,

$$\frac{1}{T}\sum_1^T \epsilon_t \widehat{u}_{t-1} \xrightarrow{p} \frac{1}{T}\sum_1^T \epsilon_t[\epsilon_{t-1} + \theta^{p+1}(\epsilon_{t-p-1} + \theta\epsilon_{t-p-2})] \xrightarrow{p} \theta\sigma^2 + O_p(T).$$

since $E(\epsilon_t\epsilon_{t-j}) = 0\ \forall j \geq 2$. If $p \longrightarrow \infty$, the second term vanishes. Thus we prove the consistency of $\widehat{\theta}$.

**(c)** Since we have derived that $(1/T)\sum_1^T \widehat{u}_{t-1}^2 \xrightarrow{p} \sigma^2[1 + \theta^{2p+2}(1+\theta^2)]$, all we need to do is check

$$\frac{1}{\sqrt{T}}\sum_1^T \epsilon_t\theta^{p+1}(\epsilon_{t-p-1} + \theta\epsilon_{t-p-2}) + \frac{1}{\sqrt{T}}\sum_1^T \epsilon_t\epsilon_{t-1}.$$

Since $(1/T)\sum_1^T \epsilon_{t-1}^2 \xrightarrow{p} \sigma^2$, $\{\epsilon_t\epsilon_{t-1}\}$ is a MDS, and $E[(\epsilon_t\epsilon_{t-1})^2] = E[E(\epsilon_t^2\epsilon_{t-1}^2)|\epsilon_{t-1}] = E(\sigma^2\epsilon_{t-1}^2) = \sigma^4$, then

$$\frac{1}{\sqrt{T}}\sum_1^T \epsilon_t\epsilon_{t-1} \xrightarrow{d} N(0, \sigma^4).$$

Note that for $j \geq 1$, we have

$$\frac{1}{\sqrt{T}}\sum_1^T \epsilon_t\epsilon_{t-j} \xrightarrow{d} N(0, \sigma^4).$$

Hence

$$\frac{1}{\sqrt{T}}\sum_1^T \theta^{p+1}\epsilon_t\epsilon_{t-p-1} \xrightarrow{d} N(0, \sigma^4\theta^{2p+2}),$$

$$\frac{1}{\sqrt{T}}\sum_1^T \theta^{p+2}\epsilon_t\epsilon_{t-p-2} \xrightarrow{d} N(0, \sigma^4\theta^{2p+4}).$$

In sum, we obtain

$$\frac{1}{\sqrt{T}}\sum_{1}^{T}\epsilon_t\epsilon_{t-1}+\frac{1}{\sqrt{T}}\sum_{1}^{T}\theta^{p+1}\epsilon_t\epsilon_{t-p-1}+\frac{1}{\sqrt{T}}\sum_{1}^{T}\theta^{p+2}\epsilon_t\epsilon_{t-p-2}$$
$$\xrightarrow{d} N(0,\sigma^4(1+\theta^{2p+2}+\theta^{2p+4})).$$

Combined with $(1/T)\sum_{1}^{T}\widehat{u}_{t-1}^2 \xrightarrow{p} \sigma^2[1+\theta^{2p+2}(1+\theta^2)]$, we can prove asymptotic normality.

The departure from the MLE case (in which $\epsilon_t$ is observed) is the higher order terms in the asymptotic variance. The efficiency of $\widehat{\theta}$ achieves $\widehat{\theta}_{MLE}$ only when $p \longrightarrow \infty$.

**4.**

**(a)** Let $Z_t = \begin{bmatrix} y_t \\ y_{t-1} \\ \epsilon_t \end{bmatrix}$, $A = \begin{bmatrix} 1.2 & -0.5 & 0.25 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $V_t = \begin{bmatrix} \epsilon_t \\ 0 \\ \epsilon_t \end{bmatrix}$, $C = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, then this process can be written as $Z_t = C + AZ_{t-1} + V_t$.

**(b)** It is easy to verify the eigenvalues of $A$ are 0 and $0.6\pm0.3742i$. Since $\sqrt{(0.6)^2+(0.3742)^2} < 1$, this process is stationary.

**(c)** Note that

$$E(Z_{t+2}-\mu|Z_{t-1}-\mu,...) = A^3(Z_{t-1}-\mu).$$

Since $A^3 = \begin{bmatrix} 0.528 & -0.47 & 0.235 \\ 0.94 & -0.6 & 0.3 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mu = 2/(1-1.2+0.5) = 20/3$, the first row of the above matrix gives

$$E(y_{t+2}-\frac{20}{3}|y_{t-1},...) = 0.528(y_{t-1}-\frac{20}{3}) - 0.47(y_{t-2}-\frac{20}{3}) + 0.235\epsilon_{t-1},$$

or

$$E(y_{t+2}|y_{t-1},...) = 6.28 + 0.528y_{t-1} - 0.47y_{t-2} + 0.235\epsilon_{t-1}.$$

**(d)** Given that we have derived the state space representation, notice that

$$y_t = \frac{20}{3} + \epsilon_t + A_{11}\epsilon_{t-1} + A_{11}^2\epsilon_{t-2} + ...,$$

where $A_{11}^j$ is the (1,1) element in the matrix $A^j$.