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PROBLEM SET 1

Instructions

Part I – Analytical Questions

1. Suppose

$$\sqrt{T}\left(\hat{\theta}_T - \theta_0\right) \xrightarrow{L} N(0, \sigma^2)$$

Does it follow then that $h(\hat{\theta}_T) \xrightarrow{p} h(\theta_0)$ where h is a continuous function? Explain your answer.

Solution

We know the result holds as long as θ_0 is a constant. This is true in this case since θ_0 is the population mean. Also, notice that since $\underset{T\to\infty}{p\lim}\frac{1}{\sqrt{T}} = 0$, and $\sqrt{T}\left(\hat{\theta}_T - \theta_0\right) \xrightarrow{L} N(0, \sigma^2)$, which is therefore bounded (or $O_p(T^{1/2})$), it immediately follows that

$$\underset{T\to\infty}{p\lim}\, \hat{\theta}_T - \theta_0 = 0$$

2. *Combining the delta method and Lindeberg-Levy.*

Let $\{z_t\}$ be an i.i.d. sequence of random variables with $E(z_t) = \mu \neq 0$ and $V(z_t) = \sigma^2$. Let $\bar{z}_T$ be the sample mean, derive the asymptotic distribution of $\ln(\bar{z}_T)$.

Solution

By Lindeberg-Levy and the conditions of the problem it is easy to see that

$$\sqrt{T}(\bar{z}_T - \mu) \xrightarrow{L} N(0, \sigma^2).$$

Next, apply the delta method. First, note that $g(x) = \ln(x)$, therefore, $g'(x_0) = 1/(x_0)$. Here, $x = \bar{z}_T$ and $x_0 = \mu$. Therefore

$$\sqrt{T}\left(\ln(\bar{z}_T) - \ln(\mu)\right) \xrightarrow{L} N\left(0, \frac{\sigma^2}{\mu^2}\right)$$

3. Calculate the variance of the process $y_t = y_{t-1} + \varepsilon_t$ where $\varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$. Hence, show that it is not stationary.

Solution

Notice that each element in the sequence $\{y_t\}$ is just the sum of all past innovations, i.e.

$$y_t = \sum_{s=0}^{t} \varepsilon_s$$

and therefore $V(y_t) = \sum_0^t \sigma^2 = t\sigma^2$ (by the independence of the ε). Hence, as t grows so does the variance of y. Because the variance is a function of time, we have shown that the process is non-stationary.

4. Suppose $X_T \xrightarrow{P} \frac{1}{\sigma}$ and $Y_T \xrightarrow{P} Y$ where $Y \sim N(0, \sigma^2)$. Derive the limiting distribution of $X_T Y_T$. Hence derive the distribution of $(X_T Y_T)^2$. Be explicit about your assumptions.

Solution

We know that convergence in probability implies convergence in distribution. Also from Slutsky's theorem we know that if $X_T \xrightarrow{P} c$ and $Y_T \xrightarrow{L} Y$ then $X_T Y_T \xrightarrow{L} cY$. Since Y is normally distributed, then $\frac{Y}{\sigma} \sim N(0,1)$. Finally, note that the square of a N(0,1) random variable is distributed chi-square and hence, $(X_T Y_Y)^2 \xrightarrow{d} \chi_1^2$

5. Does a martingale difference sequence have to be covariance stationary? Explain.

Solution

No, the variance could be a function of time.