

PROBLEM SET 5

Instructions

This problem set is divided into two parts: (1) Analytical Questions, and (2) Applied Questions.

Please try to answer the questions rigorously by stating any implied assumptions and ensuring all the steps to your conclusion have been properly verified.

Part I – Analytical Questions

Problem 1: Consider the following DGP

$$\begin{aligned}x_t + \beta y_t &= u_{1t}; & \text{where } u_{1t} &= \theta u_{1t-1} + \varepsilon_{1t} \\x_t + \alpha y_t &= u_{2t}; & \text{where } u_{2t} &= \rho u_{2t-1} + \varepsilon_{2t}\end{aligned}$$

with $|\rho| < 1$ and

$$\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \sim D \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \gamma \\ \gamma & \sigma_2^2 \end{pmatrix} \right]$$

where D denotes a generic distribution.

(a) Derive the degree of integratedness of the two series, x_t and y_t , explicitly stating the parameter restrictions required in each case.

(b) Under what coefficient restrictions are x and y cointegrated? What are the cointegrating vectors in such cases?

(c) Choose a particular set of coefficients that ensures x and y are cointegrated and derive the following representations:

(i) the moving-average: that is $(x_t \ y_t)'$ on the left hand side, $(\varepsilon_{1t} \ \varepsilon_{2t})'$ and its lags on the right hand side.

(ii) The autoregressive in the levels: that is, $(x_t \ y_t)'$ on the right hand side, and lags of $(x_t \ y_t)'$ and $(\varepsilon_{1t} \ \varepsilon_{2t})'$ on the right hand side.

(iii) The Error-Correction Representation: that is, $(\Delta x_t \ \Delta y_t)'$ as a function of z_{t-1} and residuals (no need to be specific about the residuals).

(d) [5 points] In general, discuss the pros and cons of obtaining impulse responses from a VAR estimated in the levels, as in part (c)-(i) and one in vector error correction form, as in part (c)-(iii).

Solution:

(a) If $\theta = 1$ then $x_t + \beta y_t \sim I(1)$ but $x_t + y_t \sim I(0)$. Hence $\alpha \neq \beta$ and:

- If $\alpha \neq 0$, $x_t \sim I(1)$ and $y_t \sim I(1)$ but $(1 \ \alpha)$ is a cointegrating vector.
- If $\alpha = 0$, then $x_t \sim I(0)$ and $y_t \sim I(1)$

If $|\theta| < 1$ then both x and y are stationary.

(b) $\theta = 1, \alpha \neq 0, \beta \neq 0; (1 \ \alpha)$

(c) (i)

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 1 & \alpha \end{pmatrix}^{-1} \begin{pmatrix} (1-L)^{-1} & 0 \\ 0 & (1-\rho L)^{-1} \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} =$$

$$x_t = \frac{1}{\alpha - \beta} \left(-\beta \sum \varepsilon_{1t} + \alpha \sum \rho^t \varepsilon_{2t} \right)$$

$$y_t = \frac{1}{\alpha - \beta} \left(\sum \varepsilon_{1t} - \sum \rho^t \varepsilon_{2t} \right)$$

(ii)

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \frac{1}{\alpha - \beta} \begin{pmatrix} -\beta & \alpha \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \frac{1}{\alpha - \beta} \begin{pmatrix} -\beta & \alpha \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

$$x_t = \frac{1}{\alpha - \beta} \left(-\beta x_{t-1} + \alpha \rho y_{t-1} - \beta \varepsilon_{1t} + \alpha \varepsilon_{2t} \right)$$

$$y_t = \frac{1}{\alpha - \beta} \left(x_{t-1} - \rho y_{t-1} + \varepsilon_{1t} - \varepsilon_{2t} \right)$$

(iii)

$$\Delta x_t = \frac{\beta(1-\rho)}{\alpha - \beta} z_{t-1} + \eta_{1t}$$

$$\Delta y_t = -\frac{(1-\rho)}{\alpha - \beta} z_{t-1} + \eta_{2t}$$

(d) the VAR in levels is guaranteed to be consistent but not as efficient as when we impose the cointegrating restrictions. However, imposing incorrect cointegrating

restrictions can lead to misspecification. Additionally, it is even more complicated to obtain standard errors for impulse responses from cointegrated VARs.

Problem 2: Consider the following D.G.P.

$$\begin{aligned}x_t + y_t &= v_t, & v_t(1 - \rho_1 L) &= \varepsilon_{1t} \\2x_t + y_t &= u_t, & u_t(1 - \rho_2 L) &= \varepsilon_{2t}\end{aligned}$$

and

$$\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \sim N \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \right]$$

(a) Determine whether this system is stationary, non-stationary, or non-stationary but cointegrated according to the following scenarios:

- (i) $|\rho_1| < 1, |\rho_2| < 1$
- (ii) $\rho_1 = 1, |\rho_2| < 1$
- (iii) $\rho_1 = 1, \rho_2 = 1$

Solution:

- (i) *Stationary*
- (ii) *Cointegrated*
- (iii) *Nonstationary*

(b) Obtain the reduced-form, autoregressive representation of the system in the levels when it is cointegrated.

Solution:

$$\begin{aligned}(1 - L)x_t + (1 - L)y_t &= \varepsilon_{1t} \\2(1 - \rho_2 L)x_t + (1 - \rho_2 L)y_t &= \varepsilon_{2t}\end{aligned}$$

rearranging

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2\rho_2 & \rho_2 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

Finally, the reduced form is obtained by inverting the matrix of contemporaneous correlations

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 2\rho_2 - 1 & \rho_2 - 1 \\ 2(1 - \rho_2) & 2 - \rho_2 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} -\varepsilon_{1t} + \varepsilon_{2t} \\ 2\varepsilon_{1t} - \varepsilon_{2t} \end{pmatrix}$$

(c) Given the representation that you just found, calculate the reduced-form, impulse response function coefficient matrices, ψ_s for periods $s = 0, 1, \text{ and } 2$ for $\rho_2 = 0.5$. What is ψ_s as $s \rightarrow \infty$? Given the reduced form impulse response matrices, calculate the structural impulse responses for periods 0, 1, 2. What is happening as $s \rightarrow \infty$? Explain this result.

Solution:

Reduced Form

$$\psi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\psi_1 = \begin{pmatrix} 0 & -0.5 \\ 1 & 1.5 \end{pmatrix}$$

$$\psi_2 = \begin{pmatrix} -0.5 & -0.75 \\ 1.5 & 1.75 \end{pmatrix}$$

\vdots

$$\psi_s \rightarrow \infty, s \rightarrow \infty$$

Structural

$$\psi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\psi_1 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -0.5 \\ 1 & 1.5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0.5 \end{pmatrix}$$

$$\psi_2 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -0.5 & -0.75 \\ 1.5 & 1.75 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0.5 & 0.25 \end{pmatrix}$$

\vdots

$$\psi_s \rightarrow \infty, s \rightarrow \infty \text{ but only for } x_t$$

(d) Find the moving-average representation of the system and the cointegrating vector when it is cointegrated.

Solution

$$x_t = -\frac{\varepsilon_{1t}}{(1-L)} + \frac{\varepsilon_{2t}}{(1-\rho_2 L)}$$

$$y_t = \frac{2\varepsilon_{1t}}{(1-L)} - \frac{\varepsilon_{2t}}{(1-\rho_2 L)}$$

hence the obvious cointegrating vector is $(2 \ 1)'$

(e) Describe how you could estimate the cointegrating vector in a regression of y_t on x_t which is well behaved in small samples.

Solution:

In large samples the regression of y_t on x_t will deliver an asymptotically consistent estimate of the coefficient λ in the regression

$$y_t = \lambda x_t + u_t$$

however, notice that the error term u_t is $u_t = -\frac{\varepsilon_{2t}}{(1-\rho_2L)}$. Typically this is of unknown

form so a recommended strategy to correct for small sample bias is to use the Saikkonen, Phillips and Loretan, or Stock and Watson approach of including lags and leads of Δx_t . Here, because the source of the correlation in the residuals is known, we have that an AR(1) correction would solve the problem since,

$$(1-\rho_2L)y_t = -2x_t(1-\rho_2L) - \varepsilon_{2t}$$

Problem 3: Consider the following bivariate VAR

$$y_{1t} = 0.3y_{1t-1} + 0.8y_{2t-1} + \varepsilon_{1t}$$

$$y_{2t} = 0.9y_{1t-1} + 0.4y_{2t-1} + \varepsilon_{2t}$$

with $E(\varepsilon_{1t}\varepsilon_{1\tau}) = 1$ for $t = \tau$ and 0 otherwise, $E(\varepsilon_{2t}\varepsilon_{2\tau}) = 2$ for $t = \tau$ and 0 otherwise, and $E(\varepsilon_{1t}\varepsilon_{2\tau}) = 0$ for all t , and τ . Answer the following questions:

(a) Is this system covariance-stationary?

Solution

To answer this question, verify the roots of the polynomial

$$\left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.3 & 0.8 \\ 0.9 & 0.4 \end{pmatrix} z \right| = (1 - 0.3z)(1 - 0.4z) - (0.8z)(0.9z) = 1 - 0.7z - 0.6z^2$$

The roots are 0.833 and 2, hence the system is not-stationary.

(b) Calculate $\psi_s = \frac{\partial y_{t+s}}{\partial \varepsilon_t}$ for $s = 0, 1$, and 2. What is the limit as $s \rightarrow \infty$?

Solution

$$\psi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \psi_1 = \begin{pmatrix} 0.3 & 0.8 \\ 0.9 & 0.4 \end{pmatrix}; \psi_2 = \begin{pmatrix} 0.81 & 0.56 \\ 0.63 & 0.88 \end{pmatrix}$$

Clearly, since $\psi_s = \begin{pmatrix} 0.3 & 0.8 \\ 0.9 & 0.4 \end{pmatrix}^s$ and the process is not stationary, then $\psi_s \rightarrow \infty$.

(c) Calculate the fraction of the MSE of the two period-ahead forecast error for variable 1, $E[y_{1,t+2} - \hat{E}(y_{1,t+2} | y_t, y_{t-1}, \dots)]^2$, that is due to $\varepsilon_{1,t+1}$ and $\varepsilon_{1,t+2}$

Solution

$$E[y_{1,t+2} - \hat{E}(y_{1,t+2} | y_t, y_{t-1}, \dots)]^2 = E[\varepsilon_{1,t+2} + 0.3\varepsilon_{1,t+1} + 0.8\varepsilon_{2,t+1}]^2 = 1 + 0.3^2 + 0.8^2 \times 2 = 2.37$$

The fraction due to ε_1 is $(1 + 0.32)/2.37 = 0.46$ or 46%.

Problem 4: Consider the following VAR

$$y_t = (1 + \beta)y_{t-1} - \beta\alpha x_{t-1} + \varepsilon_{1t}$$

$$x_t = \gamma y_{t-1} + (1 - \gamma\alpha)x_{t-1} + \varepsilon_{2t}$$

(a) Show that this VAR is not-stationary.

Stationarity requires that the values of z satisfying

$$\left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 + \beta & -\beta\alpha \\ \gamma & (1 - \gamma\alpha) \end{pmatrix} z \right| = 0$$

lie outside the unit circle. For $z = 1$, notice

$$\begin{vmatrix} -\beta & \beta\alpha \\ -\gamma & \gamma\alpha \end{vmatrix} = -\beta\gamma\alpha + \beta\gamma\alpha = 0$$

(b) Find the cointegrating vector and derive the VECM representation.

Notice that

$$\Phi(1) = \begin{pmatrix} -\beta & \beta\alpha \\ -\gamma & \gamma\alpha \end{pmatrix} = (-\beta \quad -\gamma)' (1 \quad -\alpha)$$

so that

$$\begin{aligned} \Delta y_t &= \beta(y_{t-1} - \alpha x_{t-1}) + \varepsilon_{1t} \\ \Delta x_t &= \gamma(y_{t-1} - \alpha x_{t-1}) + \varepsilon_{2t} \end{aligned}$$

(c) Transform the model so that it involves the error correction term (call it z) and a difference stationary variable (call it Δw_t). w will be a linear combination of x and y but should not contain z . *Hint:* the weights in this linear combination will be related the coefficients of the error correction terms.

Given the ECM in part (b), notice

$$\begin{aligned} -\gamma\Delta y_t + \beta\Delta x_t &= -\gamma\beta z_{t-1} - \gamma\varepsilon_{1t} + \beta\gamma z_{t-1} + \beta\varepsilon_{2t} \\ \Delta w_t &= -\gamma\Delta y_t + \beta\Delta x_t = -\gamma\varepsilon_{1t} + \beta\varepsilon_{2t} \end{aligned}$$

Next

$$\begin{aligned}
 y_t &= y_{t-1} + \beta(y_{t-1} - \alpha x_{t-1}) + \varepsilon_{1t} \\
 x_t &= x_{t-1} + \gamma(y_{t-1} - \alpha x_{t-1}) + \varepsilon_{2t} \\
 (y_t - \alpha x_t) &= (y_{t-1} - \alpha x_{t-1}) + \beta z_{t-1} - \alpha \gamma z_{t-1} + \varepsilon_{1t} - \alpha \varepsilon_{2t} \\
 z_t &= (1 + \beta - \alpha \gamma) z_{t-1} + \varepsilon_{1t} - \alpha \varepsilon_{2t} \\
 \begin{pmatrix} z_t \\ \Delta w_t \end{pmatrix} &= \begin{pmatrix} 1 + \beta - \alpha \gamma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_{t-1} \\ \Delta w_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & -\alpha \\ -\gamma & \beta \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}
 \end{aligned}$$

(d) Verify that y and x can be expressed as a linear combination of w and z . Give an interpretation as a decomposition of the vector $(y \ x)'$ into *permanent* and *transitory* components.

From part (c)

$$\begin{pmatrix} z_t \\ w_t \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ -\gamma & \beta \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

taking the inverse

$$\frac{1}{\beta - \alpha \gamma} \begin{pmatrix} \beta & \alpha \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} z_t \\ w_t \end{pmatrix} = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

and therefore

$$\begin{aligned}
 y_t &= \frac{\beta}{\beta - \alpha \gamma} z_t + \frac{\alpha}{\beta - \alpha \gamma} w_t \\
 x_t &= \frac{\gamma}{\beta - \alpha \gamma} z_t + \frac{1}{\beta - \alpha \gamma} w_t
 \end{aligned}$$

w_t is $I(1)$ and z_t is $I(0)$, which is a version of the Beveridge-Nelson decomposition proposed by Gonzalo and Granger (1995).

Problem 5: Consider the bivariate VECM

$$\Delta y_t = c + \alpha \beta' y_{t-1} + \varepsilon_t \quad \varepsilon_{it} \stackrel{iid}{\sim} (0, \sigma^2)$$

where $\alpha = (\alpha_1, 0)'$ and $\beta = (1, -\beta_2)'$. Equation by equation, the system is given by

$$\begin{aligned}\Delta y_{1t} &= c_1 + \alpha_1(y_{1t-1} - \beta_2 y_{2t-1}) + \varepsilon_{1t} \\ \Delta y_{2t} &= c_2 + \varepsilon_{2t}\end{aligned}$$

Answer the following questions:

- (a) From the VECM representation above, derive the VECM representation

$$\Delta y_t = c + \Pi y_{t-1} + \varepsilon_t$$

and the VAR(1) representation

$$y_t = c + A y_{t-1} + \varepsilon_t$$

$$\Pi = \begin{pmatrix} \alpha_1 & -\alpha_1 \beta_2 \\ 0 & 0 \end{pmatrix}; A = \begin{pmatrix} \alpha_1 + 1 & -\alpha_1 \beta_2 \\ 0 & 1 \end{pmatrix}$$

- (b) Based on the given values of the elements in α and β , determine $\alpha_\perp, \beta_\perp$, such that $\alpha' \alpha_\perp = 0$ and $\beta' \beta_\perp = 0$.

$$\alpha_\perp = \begin{pmatrix} 0 \\ k \end{pmatrix}; \beta_\perp = \begin{pmatrix} \beta_2 k \\ k \end{pmatrix}; k \neq 0$$

- (c) Using the Granger representation theorem determine that $\psi(1) = \beta_\perp (\alpha_\perp' I_2 \beta_\perp')^{-1} \alpha_\perp$, where $\psi(L)$ is the moving average polynomial corresponding to the VECM system above and I_2 is the identity matrix of order 2. *Hint:* you may show this result by showing that $\psi(1)$ is orthogonal to the cointegrating space.

Using the hint: $\Pi \psi(1)' = 0$. It is easy to show that

$$\beta_\perp (\alpha_\perp' I_2 \beta_\perp')^{-1} \alpha_\perp = \begin{pmatrix} 0 & \beta_2 \\ 0 & 1 \end{pmatrix}$$

and therefore

$$\begin{pmatrix} \alpha_1 & -\alpha_1 \beta_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta_2 \\ 0 & 1 \end{pmatrix} = 0$$

- (d) Using the Beveridge-Nelson decomposition and the result in (c), determine the common trend in the VECM system.

All you need to remember is that from the B-N decomposition, the trends are the linear combinations captured in $\psi(I)y_t$, which in this case turns out to be $\beta_2 y_{1t} + y_{2t}$. Notice that this combination is orthogonal to the cointegrating vector.

(e) Show that $\beta' y_t$ follows an AR(1) process and show that this AR(1) is stable provided that $-2 < \alpha_1 < 0$. What can you say about the system when $\alpha_1 = 0$?

Let $z_t = y_{1t} - \beta_2 y_{2t}$ be the cointegrating vector. From the equations for y_1 and y_2 we have

$$z_t = c_1 + (\alpha_1 + 1)y_{1t-1} - \alpha_1 \beta_2 y_{2t-1} + \varepsilon_{1t} - \beta_2 c_2 - \beta_2 y_{2t-1} - \beta_2 \varepsilon_{2t}$$

Combining terms

$$z_t = (c_1 - \beta_2 c_2) + (\alpha_1 + 1)z_{t-1} + v_t; v_t = \varepsilon_{1t} - \beta_2 \varepsilon_{2t}$$

which is an AR(1) whose stationarity requires that $|\alpha_1 + 1| < 1$ or the equivalent condition $-2 < \alpha_1 < 0$. When $\alpha_1 = 0$, z_t is no longer stationary, so there is no cointegration for any value of β_2 . y_1 and y_2 are in this case two independent random walks.