PROBLEM SET 4 – DUE: FEBRUARY 25

Instructions

This problem set is divided into two parts: (1) Analytical Questions, and (2) Applied Questions.

Please try to answer the questions rigorously by stating any implied assumptions and ensuring all the steps to your conclusion have been properly verified.

Part I – Analytical Questions

Problem 1: Let

$$y_t = \beta t^{\alpha} + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma^2)$$

and α known. Given the normality of ε_t, the exact distribution of $\hat{\beta}_{OLS}$ can be obtained. Answer the following questions:

(a) What is the distribution of $\hat{\beta}_{OLS}$ for a generic α?
(b) What is the distribution of $\hat{\beta}_{OLS}$ for $\alpha = 0$?
(c) What is the distribution of $\hat{\beta}_{OLS}$ for $\alpha = ½$? How can you rescale the problem to obtain this distribution?
(d) What is the distribution of $\hat{\beta}_{OLS}$ for $\alpha = -1$? Hint: $\lim_{T\to\infty} \sum t^{-2} = \pi^2/6$

Problem 2:

Let the true D.G.P. be

$$y_t = y_{t-1} + u_t$$

where u_t is a MDS with variance σ_u^2 and also $\sup_t E|u_t|^{2+\delta} < \infty$ for some $\delta > 0$. Also, assume that $y_0 = 0$. Consider the regression

$$y_t = X_t \beta + \varepsilon_t \quad \text{where} \quad X_t = (1 \ \ t)'.$$

Denote $\hat{\beta}$ the OLS estimator, show that

$$\begin{bmatrix} T^{-1/2}(\hat{\beta}_1 - \beta_1) \\ T^{1/2}(\hat{\beta}_2 - \beta_2) \end{bmatrix} \xrightarrow{L} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}^{-1} \begin{bmatrix} \int_0^1 W(\lambda)d\lambda \\ \int_0^1 \lambda W(\lambda)d\lambda \end{bmatrix} \sigma_u$$

Hint: the problem is easier to attack by considering the following

$$T^{-1}D_T(\hat{\beta}-\beta) = \left[D_T^{-1}\left(\sum X_t X_t'\right)D_T^{-1}\right]^{-1}\left[T^{-1}D_T^{-1}\sum X_t v_t\right]$$

where $v_t = \sum_{s=1}^{t} u_t$ and

$$D_T \begin{bmatrix} T^{1/2} & 0 \\ 0 & T^{3/2} \end{bmatrix}$$

Problem 3:
Show that if

$$y_t = \alpha + \rho y_{t-1} + u_t \quad \text{with}$$
$$u_t = \varphi_1 u_{t-1} + \varphi_2 u_{t-2} + \varphi_3 u_{t-3} + \varepsilon_t$$

then y_t can be expressed as

$$y_t = \mu + \beta y_{t-1} + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \phi_3 \Delta y_{t-3} + \varepsilon_t$$

Find the values for ϕ_1, ϕ_2, and ϕ_3, μ and β.

Problem 4:

The Sargan-Bhargawa (1983) statistic for a sample $\{y_0, \ldots, y_T\}$ is defined as

$$SB = \frac{\frac{1}{T^2}\sum_{t=0}^{T} y_t^2}{\frac{1}{T}\sum_{t=1}^{T} \Delta y_t^2}$$

(which incidentally, is the reciprocal of the Durbin-Watson statistic). Show that if $\{y_t\}$ is a driftless random walk, then

$$SB \xrightarrow{L} \int_0^1 W(r)^2 dr$$

Hint:

$$\sum_{t=0}^{T} y_t^2 = \sum_{t=1}^{T} y_t^2 + y_T^2 \quad \text{and} \quad \frac{y_T^2}{T^2} \xrightarrow{p} 0$$

Finally, what effect, if any, will serial correlation in the residuals have on the distribution of the *SB* statistic?

Problem 5:

Let $\{y_t\}$ be generated for t = 1, …, T by the process

$$y_t = \mu t + S_t \quad \text{where } S_t = \sum_{j=1}^{t} v_j \quad \text{and } v_t \overset{iid}{\sim} N(0,\sigma^2);\; S_0 = 0$$

Consider estimating by least-squares the parameters of the model

$$y_t = \mu + \rho y_{t-1} + v_t$$

Define the scaling matrix

$$C_T = \begin{pmatrix} T^{1/2} & 0 \\ 0 & T^{3/2} \end{pmatrix}$$

then show that

$$C_T\begin{pmatrix} \hat{\mu}-\mu \\ \hat{\rho}-1 \end{pmatrix} = \begin{pmatrix} 1 & T^{-2}\sum_{t=1}^{T} y_{t-1} \\ T^{-2}\sum_{t=1}^{T} y_{t-1} & T^{-3}\sum_{t=1}^{T} y_{t-1}^2 \end{pmatrix}^{-1}\begin{pmatrix} T^{-1/2}\sum_{t=1}^{T} v_t \\ T^{-3/2}\sum_{t=1}^{T} y_{t-1}v_t \end{pmatrix}$$
$$= B_T^{-1}\begin{pmatrix} T^{-1/2}\sum_{t=1}^{T} v_t \\ T^{-3/2}\sum_{t=1}^{T} y_{t-1}v_t \end{pmatrix}$$

(b) Given that

$$T^{-2}\sum_{t=1}^{T} S_t \xrightarrow{p} 0; \qquad T^{-3/2}\sum_{t=1}^{T} S_{t-1}v_t \xrightarrow{p} 0$$
$$T^{-5/2}\sum_{t=1}^{T} tS_t \xrightarrow{L} W_1 \qquad T^{-3}\sum_{t=1}^{T} S_t^{\,2} \xrightarrow{L} W_2$$

where W_1 and W_2 are non-degenerate distributions, show that:

$$\underset{T\to\infty}{p\lim}\, B_T = \begin{pmatrix} 1 & \frac{1}{2}\mu \\ \frac{1}{2}\mu & \frac{1}{3}\mu^2 \end{pmatrix} = B \quad \text{and} \quad T^{-3/2}\sum_{t=1}^{T} y_{t-1}v_t \xrightarrow{L} N(0,\frac{1}{3}\sigma^2\mu^2)$$

(c) Since the asymptotic covariance between $T^{-1/2}\sum_{t=1}^{T} v_t$ and $T^{-3/2}\sum_{t=1}^{T} y_{t-1}v_t$ is $\frac{1}{2}\sigma^2\mu$ show that:

$$\begin{pmatrix} T^{-1/2}\sum_{t=1}^{T} v_t \\ T^{-3/2}\sum_{t=1}^{T} y_{t-1}v_t \end{pmatrix} \xrightarrow{L} N\left(\begin{matrix}0\\0\end{matrix}; \sigma^2 B\right) \text{ and therefore}$$

$$C_T\begin{pmatrix} \hat{\mu}-\mu \\ \hat{\rho}-1 \end{pmatrix} \xrightarrow{L} N\left(\begin{matrix}0\\0\end{matrix}; \sigma^2 B^{-1}\right)$$

Carefully state any theorems and assumptions you make.

Hints:

$S_t = S_{t-1} + v_t$. Also

$$T^{-1}\sum_{t=1}^{T}\left(\frac{S_{t-1}}{\sqrt{T}}\right) \xrightarrow{L} \int_0^1 W(r)dr \text{ so } T^{-1}\sum_{t=1}^{T}\left(\frac{S_{t-1}}{\sqrt{T}}\right)^2 \xrightarrow{L} \int_0^1 W^2(r)dr$$

$$\lim_{T\to\infty} T^{-(n+1)}\sum_{t=1}^{T} t^n = (n+1)^{-1}$$

$$T^{-1/2}\sum_{t=1}^{T} v_t \xrightarrow{L} \int_0^1 dW(r) = W(1) \sim N(0,1)$$

$$T^{-3/2}\sum_{t=1}^{T} tv_t \xrightarrow{L} \int_0^1 rdW(r) \sim N(0,\frac{1}{3})$$

$$T^{-1}\sum_{t=1}^{T} S_{t-1}v_t \xrightarrow{L} \int_0^1 W(r)dW(r) \text{ so that } T^{-3/2}\sum_{t=1}^{T} S_{t-1}v_t \xrightarrow{p} 0$$

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1}v_t = T^{-3/2}\sum_{t=1}^{T}\mu(t-1)v_t + T^{-3/2}\sum_{t=1}^{T} S_{t-1}v_t \xrightarrow{L} \mu\int_0^1 rdW(r)$$

Alternatively, by conventional methods,

$$\begin{pmatrix} T^{-1/2}\sum_{t=1}^{T} v_t \\ T^{-3/2}\sum_{t=1}^{T} tv_t \end{pmatrix} \xrightarrow{L} \begin{pmatrix} T^{-1/2}\sum_{t=1}^{T} v_t \\ \mu T^{-3/2}\sum_{t=1}^{T} tv_t \end{pmatrix} \xrightarrow{L} N\left(\begin{matrix}0\\0\end{matrix}; \sigma^2 B\right)$$

II Applied Questions

Problem 1: Empirical Properties of the t-test near unit roots

Consider the following D.G.P.

$$y_t = \beta y_{t-1} + e_t \qquad e_t \sim N(0,1) \qquad y_0 = 0$$

where β can take on values: (i) β = 1.0; (ii) β = 0.9; (iii) β = 0.5. We will investigate the distributional properties of $\hat{\beta}_{OLS}$ with a Monte Carlo experiment. Let M = 1,000 be the number of replications in the Monte Carlo. Therefore, you need to generate 1,000 replications of series generated by the D.G.P. presented above, for each value of β with a final sample size T = 50 where you disregard the first 100 observations of each series. Thus, you will have 1,000 samples of size T = 50 for each possible value of β. You will need to estimate 1,000 OLS regressions for each value of β and calculate the following magnitudes:

(1) The asymptotic bias: $\frac{1}{M}\sum_{r=1}^{M}(\hat{\beta}_{OLS} - \beta)$

(2) The Monte Carlo standard error of $\hat{\beta}_{OLS}$, that is, $\hat{\sigma}_{\hat{\beta}} = \sqrt{\frac{\sum_{r=1}^{M}(\hat{\beta}_{OLS} - \bar{\hat{\beta}}_{OLS})}{M-1}}$

(3) The Monte Carlo average fit: $\frac{1}{M}\sum_{r=1}^{M}R^2$

(4) The size and the power of the usual t-test for the following nulls:

- (a) H_0: β = 1. That is, calculate the size by checking the rejection frequency of this test when the true model is with β = 1, and check the power by calculating the rejection frequency when β = 0.9, and 0.5.
- (b) H_0: β = 0.9
- (c) H_0: β = 0.5

Comment on the power and size distortions of the test for each null hypothesis.

(5) Plot a histogram for $\hat{\beta}_{OLS}$ for each value of β = 1, 0.9, 0.5

Comments:

- Turn in the GAUSS code you write to do this exercise.
- In a table, report the results to questions (1)-(3) with appropriate comments (no more than one paragraph).

- Turn in a table for the results in (4) with comments.
- Turn in one page with the plot of the 3 histograms described in (5).
- No handwriting please.

Problem 2: Nonsense Regressions

Consider a Monte Carlo exercise with the following D.G.P.s

$$y_t = \rho y_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim N(0,1) \qquad y_0 = 0$$
$$x_t = \rho x_{t-1} + u_t \qquad u_t \sim N(0,1) \qquad x_0 = 0$$

with $E(\varepsilon_t, u_s) = 0$ for any t and s. Let $T = 100$ (after disregarding the first 100 observations) and consider three possible values of ρ: (i) 1, (ii) 0.5, (iii) 0. You will need to do 1,000 replications for each value of ρ and then estimate the following OLS regression:

$$y_t = \beta_0 + \beta_1 x_t + v_t$$

from which you should calculate the following:

(1) Asymptotic biases for β_0 and β_1: $\frac{1}{M}\sum_{r=1}^{M}(\hat{\beta}_i - \beta_i)$, i = 0, 1. Notice that $\beta_i = 0$ given the DGP.

(2) Empirical sizes for the t-statistic for each of the following null hypotheses:

- H_0: $\beta_0 = 0$
- H_0: $\beta_1 = 0$

(3) Display the histogram for $\hat{\beta}_0$ and $\hat{\beta}_1$ for each value of ρ.

Comments:

Follow similar protocols as in the previous problem.