PROBLEM SET 4 – SOLUTIONS

Part I – Analytical Questions

Problem 1: Let

$$y_t = \beta t^\alpha + \varepsilon_t \qquad \varepsilon_t \sim N(0,\sigma^2)$$

and α known. Given the normality of ε_t, the exact distribution of $\hat{\beta}_{OLS}$ can be obtained. Answer the following questions:

(a) What is the distribution of $\hat{\beta}_{OLS}$ for a generic α?
(b) What is the distribution of $\hat{\beta}_{OLS}$ for $\alpha = 0$?
(c) What is the distribution of $\hat{\beta}_{OLS}$ for α = ½? How can you rescale the problem to obtain this distribution?
(d) What is the distribution of $\hat{\beta}_{OLS}$ for $\alpha = -1$? Hint: $\lim_{T\to\infty}\sum t^{-2} = \pi^2/6$

Solutions:
(a)

$$\hat{\beta}_{OLS} - \beta = \left(\sum_{t=1}^{T} t^{2\alpha}\right)^{-1}\sum_{t=1}^{T} t^{\alpha}\varepsilon_t \quad \text{so that}$$

$$E\left(\hat{\beta}_{OLS} - \beta\right) = 0; \text{ and } E\left[\left(\hat{\beta}_{OLS} - \beta\right)^2\right] = \sigma^2\left(\sum_{t=1}^{T} t^{2\alpha}\right)^{-1}$$

and

$$\hat{\beta}_{OLS} \sim N\left(\beta, \sigma^2\left(\sum_{t=1}^{T} t^{2\alpha}\right)^{-1}\right)$$

(b)

$$\sqrt{T}\left(\hat{\beta}_{OLS} - \beta\right) \sim N(0,\sigma^2)$$

(c)

Note:

$$\sum_{t=1}^{T} t^{2\alpha} = \sum_{t=1}^{T} t = \frac{T(T+1)}{2} \sim O(T^2)$$

Thus:

$$\underset{T\to\infty}{p\lim}\left(\hat{\beta}_{OLS} - \beta\right) = \underset{T\to\infty}{p\lim}\sqrt{T}\left(\hat{\beta}_{OLS} - \beta\right) = 0$$

However:

$$T(\beta_{OLS}-\beta) \sim N\left(0, \frac{2\sigma^2}{(1+T^{-1})}\right) \to N(0, 2\sigma^2)$$

(d)

$$\left(\hat{\beta}_{OLS}-\beta\right) \xrightarrow{L} N\left(0, \frac{6\sigma^2}{\pi^2}\right)$$

Thus, $\hat{\beta}_{OLS}$ is unbiased but inconsistent for β since the variance does not go to zero as the sample size goes to infinity.

Problem 2:

Let the true D.G.P. be

$$y_t = y_{t-1} + u_t$$

where u_t is a MDS with variance σ_u^2 and also $\sup_t E|u_t|^{2+\delta} < \infty$ for some $\delta > 0$. Also, assume that $y_0 = 0$. Consider the regression

$$y_t = X_t \beta + \varepsilon_t \quad \text{where} \quad X_t = (1 \;\; t)'.$$

Denote $\hat{\beta}$ the OLS estimator, show that

$$\begin{bmatrix} T^{-1/2}(\hat{\beta}_1 - \beta_1) \\ T^{1/2}(\hat{\beta}_2 - \beta_2) \end{bmatrix} \xrightarrow{L} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}^{-1} \begin{bmatrix} \int_0^1 W(\lambda)d\lambda \\ \int_0^1 \lambda W(\lambda)d\lambda \end{bmatrix} \sigma_u$$

Hint: the problem is easier to attack by considering the following

$$T^{-1}D_T(\hat{\beta}-\beta) = \left[D_T^{-1}\left(\sum X_t X_t'\right)D_T^{-1}\right]^{-1}\left[T^{-1}D_T^{-1}\sum X_t v_t\right]$$

where $v_t = \sum_{s=1}^{t} u_t$ and

$$D_T \begin{bmatrix} T^{1/2} & 0 \\ 0 & T^{3/2} \end{bmatrix}$$

Solution:

$$T^{-1}D_T(\hat{\beta}-\beta) = \left[D_T^{-1}\sum X_t X_t' D_T^{-1}\right]^{-1}\left[T^{-1}D_T^{-1}\sum X_t v_t\right]$$

$$D_T^{-1}\sum X_t X_t' D_T^{-1} = \begin{bmatrix} \frac{1}{\sqrt{T}} & 0 \\ 0 & \frac{1}{T^{3/2}} \end{bmatrix} \begin{bmatrix} (T-1) & \sum t \\ \sum t & \sum t^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{T}} & 0 \\ 0 & \frac{1}{T^{3/2}} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{T-1}{T} & \frac{1}{T^2}\sum t \\ \frac{1}{T^2}\sum t & \frac{1}{T^3}\sum t^2 \end{bmatrix} \xrightarrow{p} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix};$$

$$T^{-1}D_T^{-1}\sum X_t v_t = \begin{bmatrix} \frac{1}{T^{3/2}}\sum v_t \\ \frac{1}{T^{5/2}}\sum t v_t \end{bmatrix} \xrightarrow{L} \begin{bmatrix} \sigma_u \int W(\lambda)d\lambda \\ \sigma_u \int \lambda W(\lambda)d\lambda \end{bmatrix}$$

Putting things together replicates the desired result.

<u>Problem 3:</u>
Show that if

$$y_t = \alpha + \rho y_{t-1} + u_t \quad \text{with}$$
$$u_t = \varphi_1 u_{t-1} + \varphi_2 u_{t-2} + \varphi_3 u_{t-3} + \varepsilon_t$$

then y_t can be expressed as

$$y_t = \mu + \beta y_{t-1} + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \phi_3 \Delta y_{t-3} + \varepsilon_t$$

Find the values for ϕ_1, ϕ_2, and ϕ_3, μ and β.

<u>Solution:</u>

Rewrite the process of y_t by incorporating the dynamics of the error term as

$$(1-\varphi_1 L - \varphi_2 L^2 - \varphi_3 L^3) y_t = \alpha(1-\varphi_1-\varphi_2-\varphi_3) + \rho(1-\varphi_1 L - \varphi_2 L^2 - \varphi_3 L^3) y_{t-1} + \varepsilon_t \quad \text{or}$$
$$y_t = \alpha(1-\varphi_1-\varphi_2-\varphi_3) + (\varphi_1+\rho) y_{t-1} + (\varphi_2 - \rho\varphi_1) y_{t-2} + (\varphi_3 - \rho\varphi_2) y_{t-3} - \rho\varphi_4 y_{t-4} + \varepsilon_t$$

Let

$$\begin{aligned}
\gamma_1 &= (\varphi_1 + \rho) \\
\gamma_2 &= (\varphi_2 - \rho\varphi_1) \\
\gamma_3 &= (\varphi_3 - \rho\varphi_2) \\
\gamma_4 &= -\rho\varphi_4
\end{aligned}$$

Then from the Beveridge-Nelson decomposition, we have the following correspondence between parameters

$$(1-\beta L) - (\phi_1 L + \phi_2 L^2 + \phi_3 L^3)(1-L) = \gamma_1 L + \gamma_2 L^2 + \gamma_3 L^3 + \gamma_4 L^4$$

and

$$\begin{aligned}
\beta &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \\
\phi_1 &= -[\gamma_2 + \gamma_3 + \gamma_4] \\
\phi_2 &= -[\gamma_3 + \gamma_4] \\
\phi_3 &= -\gamma_4
\end{aligned}$$

so that

$$\begin{aligned}
\mu &= \alpha(1-\varphi_1 - \varphi_2 - \varphi_3) \\
\beta &= \rho + (1-\rho)\varphi_1 + (1-\rho)\varphi_2 + \varphi_3 - \rho\varphi_4 \\
\phi_1 &= \rho\varphi_1 + \varphi_2(\rho - 1) - \varphi_3 + \rho\varphi_4 \\
\phi_2 &= \rho\varphi_2 - \varphi_3 + \rho\varphi_4 \\
\phi_3 &= \rho\varphi_4
\end{aligned}$$

<u>Problem 4:</u>

The Sargan-Bhargawa (1983) statistic for a sample $\{y_0, \ldots, y_T\}$ is defined as

$$SB = \frac{\frac{1}{T^2}\sum_{t=0}^{T} y_t^2}{\frac{1}{T}\sum_{t=1}^{T} \Delta y_t^2}$$

(which incidentally, is the reciprocal of the Durbin-Watson statistic). Show that if $\{y_t\}$ is a driftless random walk, then

$$SB \xrightarrow{L} \int_0^1 W(r)^2 dr$$

Hint:

$$\sum_{t=0}^{T} y_t^2 = \sum_{t=1}^{T} y_t^2 + y_T^2 \text{ and } \frac{y_T^2}{T^2} \xrightarrow{p} 0$$

Finally, what effect, if any, will serial correlation in the residuals have on the distribution of the *SB* statistic?

<u>Solution:</u>

First, consider the numerator. Using the hint, all we need to do is find the distribution of

$$\frac{1}{T^2}\sum_{t=1}^{T} y_t^2 = \frac{1}{T^2}\sum_{t=1}^{T}\left(\sum_{s=1}^{t} e_s\right)^2$$

From the FCLT we know

$$\frac{1}{\sigma\sqrt{T}}\sum_{s=1}^{t} e_s \xrightarrow{L} W(.)$$

Therefore, using the FCLT and the CMT, notice

$$\frac{1}{T^2}\sum_{t=1}^{T} y_t^2 = \frac{1}{T^2}\frac{\sigma^2}{T}\sum_{t=1}^{T}\left(\frac{1}{\sigma\sqrt{T}}\sum_{s=1}^{t} e_s\right)^2 \xrightarrow{L} \sigma^2\int_0^1 W(r)^2 dr$$

Next, consider the denominator. Notice that $\Delta y_t = e_t$, hence, it is trivial to show that

$$\frac{1}{T}\sum_{t=1}^{T} \Delta y_t^2 \xrightarrow{p} \sigma^2$$

Putting these two results together completes the proof. Note that is the residuals were not white noise but they are covariance stationary, then we need to apply the FCLT for dependent processes instead. Hence

$$\frac{1}{T^2}\sum_{t=1}^{T} y_t^2 \xrightarrow{L} \varphi(1)^2\sigma^2\int_0^1 W(r)^2 dr$$

However, notice that the denominator will now converge in probability to

$$\frac{1}{T}\sum_{t=1}^{T} \Delta y_t^2 \xrightarrow{p} \gamma_0$$

so that the limiting distribution becomes

$$SB \xrightarrow{L} \frac{\varphi(1)^2 \sigma^2}{\gamma_0} \int_0^1 W(r)^2 dr$$

<u>Problem 5:</u>

Let $\{y_t\}$ be generated for $t = 1, \ldots, T$ by the process

$$y_t = \mu t + S_t \quad \text{where } S_t = \sum_{j=1}^{t} v_j \quad \text{and } v_t \overset{iid}{\sim} N(0, \sigma^2);\ S_0 = 0$$

Consider estimating by least-squares the parameters of the model

$$y_t = \mu + \rho y_{t-1} + v_t$$

Define the scaling matrix

$$C_T = \begin{pmatrix} T^{1/2} & 0 \\ 0 & T^{3/2} \end{pmatrix}$$

then show that

$$C_T \begin{pmatrix} \hat{\mu} - \mu \\ \hat{\rho} - 1 \end{pmatrix} = \begin{pmatrix} 1 & T^{-2} \sum_{t=1}^{T} y_{t-1} \\ T^{-2} \sum_{t=1}^{T} y_{t-1} & T^{-3} \sum_{t=1}^{T} y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} T^{-1/2} \sum_{t=1}^{T} v_t \\ T^{-3/2} \sum_{t=1}^{T} y_{t-1} v_t \end{pmatrix}$$

$$= B_T^{-1} \begin{pmatrix} T^{-1/2} \sum_{t=1}^{T} v_t \\ T^{-3/2} \sum_{t=1}^{T} y_{t-1} v_t \end{pmatrix}$$

(b) Given that

$$T^{-2} \sum_{t=1}^{T} S_t \xrightarrow{p} 0; \qquad T^{-3/2} \sum_{t=1}^{T} S_{t-1} v_t \xrightarrow{p} 0$$
$$T^{-5/2} \sum_{t=1}^{T} t S_t \xrightarrow{L} W_1 \qquad T^{-3} \sum_{t=1}^{T} S_t^{\,2} \xrightarrow{L} W_2$$

where W_1 and W_2 are non-degenerate distributions, show that:

$$\underset{T \to \infty}{p\lim}\, B_T = \begin{pmatrix} 1 & \frac{1}{2}\mu \\ \frac{1}{2}\mu & \frac{1}{3}\mu^2 \end{pmatrix} = B \quad \text{and} \quad T^{-3/2} \sum_{t=1}^{T} y_{t-1} v_t \xrightarrow{L} N(0, \frac{1}{3}\sigma^2 \mu^2)$$

(c) Since the asymptotic covariance between $T^{-1/2}\sum_{t=1}^{T} v_t$ and $T^{-3/2}\sum_{t=1}^{T} y_{t-1}v_t$ is $\frac{1}{2}\sigma^2\mu$ show that:

$$\begin{pmatrix} T^{-1/2}\sum_{t=1}^{T} v_t \\ T^{-3/2}\sum_{t=1}^{T} y_{t-1}v_t \end{pmatrix} \xrightarrow{L} N\left(\begin{matrix}0\\0\end{matrix}; \sigma^2 B\right) \text{ and therefore}$$

$$C_T\begin{pmatrix} \hat{\mu}-\mu \\ \hat{\rho}-1 \end{pmatrix} \xrightarrow{L} N\left(\begin{matrix}0\\0\end{matrix}; \sigma^2 B^{-1}\right)$$

Carefully state any theorems and assumptions you make.

Hints:

$S_t = S_{t-1} + v_t$. Also

$$T^{-1}\sum_{t=1}^{T}\left(\frac{S_{t-1}}{\sqrt{T}}\right) \xrightarrow{L} \int_0^1 W(r)dr \text{ so } T^{-1}\sum_{t=1}^{T}\left(\frac{S_{t-1}}{\sqrt{T}}\right)^2 \xrightarrow{L} \int_0^1 W^2(r)dr$$

$$\lim_{T\to\infty} T^{-(n+1)}\sum_{t=1}^{T} t^n = (n+1)^{-1}$$

$$T^{-1/2}\sum_{t=1}^{T} v_t \xrightarrow{L} \int_0^1 dW(r) = W(1) \sim N(0,1)$$

$$T^{-3/2}\sum_{t=1}^{T} t v_t \xrightarrow{L} \int_0^1 r\,dW(r) \sim N(0,\frac{1}{3})$$

$$T^{-1}\sum_{t=1}^{T} S_{t-1}v_t \xrightarrow{L} \int_0^1 W(r)dW(r) \text{ so that } T^{-3/2}\sum_{t=1}^{T} S_{t-1}v_t \xrightarrow{p} 0$$

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1}v_t = T^{-3/2}\sum_{t=1}^{T}\mu(t-1)v_t + T^{-3/2}\sum_{t=1}^{T} S_{t-1}v_t \xrightarrow{L} \mu\int_0^1 r\,dW(r)$$

Alternatively, by conventional methods,

$$\begin{pmatrix} T^{-1/2}\sum_{t=1}^{T} v_t \\ T^{-3/2}\sum_{t=1}^{T} t v_t \end{pmatrix} \xrightarrow{L} \begin{pmatrix} T^{-1/2}\sum_{t=1}^{T} v_t \\ \mu T^{-3/2}\sum_{t=1}^{T} t v_t \end{pmatrix} \xrightarrow{L} N\left(\begin{matrix}0\\0\end{matrix}; \sigma^2 B\right)$$

Solution:

(a) Define the vector $x_t = (1\ y_{t-1})$' then notice that

$$\begin{pmatrix} \hat{\mu}-\mu \\ \hat{\rho}-\rho \end{pmatrix} = \begin{pmatrix} T & \sum_{t=1}^{T} y_{t-1} \\ \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^{T} v_t \\ \sum_{t=1}^{T} y_{t-1} v_t \end{pmatrix}$$

and that

$$C_T \begin{pmatrix} \hat{\mu}-\mu \\ \hat{\rho}-1 \end{pmatrix} = \left\{ C_T^{-1} \begin{pmatrix} T & \sum_{t=1}^{T} y_{t-1} \\ \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-1}^2 \end{pmatrix} C_T^{-1} \right\}^{-1} C_T^{-1} \begin{pmatrix} \sum_{t=1}^{T} v_t \\ \sum_{t=1}^{T} y_{t-1} v_t \end{pmatrix}$$

(b) Calculating the p*lim* of each element of B_T, we have

$$\underset{T\to\infty}{p\lim} 1 = 1$$

$$\underset{T\to\infty}{p\lim} \frac{\sum_{t=1}^{T} y_{t-1}}{T^2} = \underset{T\to\infty}{p\lim} \frac{\sum \mu t}{T^2} + \frac{\sum S_t}{T^2}. \text{ Note that } \frac{\sum \mu t}{T^2} = \frac{\mu}{2}\frac{(T+1)(T+2)}{T}$$

and using the hint, $\underset{T\to\infty}{p\lim} \frac{\sum S_t}{T^2} = 0$ it is trivial to show $\underset{T\to\infty}{p\lim} \frac{\sum_{t=1}^{T} y_{t-1}}{T^2} = \frac{\mu}{2}$

$$\underset{T\to\infty}{p\lim} \frac{\sum_{t=1}^{T} y_{t-1}^2}{T^3} = \underset{T\to\infty}{p\lim} \left(\frac{\sum_{t=0}^{T-1} \left(\mu^2 t^2 + 2\mu t S_t + S_t^2\right)}{T^3} \right).$$

Note:

$\underset{T\to\infty}{p\lim} 2\mu \frac{\sum t S_t}{T^3} = 0$ since the hint tells us $T^{-5/2} \sum t S_t \xrightarrow{L} W_1$

$$\underset{T\to\infty}{p\lim} \frac{\mu^2 \sum t^2}{T^3} = \underset{T\to\infty}{p\lim} \frac{\mu^2}{6} \frac{(T+1)(T+2)[2(T+1)+1]}{T^3} = \frac{\mu^2}{3}$$

Finally, the hint also tells us that $\frac{\sum S_t^2}{T^3} \xrightarrow{L} W_2$. It is not obvious to me how we get from this result to $\frac{\sum S_t^2}{T^3} \xrightarrow{p} 0$, which would allow us to conclude the proof.

To proof the last result, note

$T^{-3/2} \sum y_{t-1} v_t = T^{-3/2} \left(\mu \sum (t-1) v_t + \sum S_{t-1} v_t \right)$, therefore, using the hints it is trivial to show $T^{-3/2} \sum y_{t-1} v_t \xrightarrow{L} N\left(0, \frac{1}{3}\sigma^2 \mu^2\right)$

(c) The last part really consists on putting the elements in parts (a) and (b) together.