1. CH.1: Introduction

1-1 (a) $p(\mathbf{y}|\theta)=\prod_{i-1}^{N}p(\mathbf{y}_i|\theta)=\prod_{i-1}^{N}\theta e^{-\theta y_i}=\theta^N e^{-\theta\sum_i y_i}=\theta^N e^{-\theta N\bar{y}}$.

(b) MLE minimizes ln$p(\mathbf{y}|\theta)=N\ln\theta-\theta N\bar{y}$.
$\partial\ln p(\mathbf{y}|\theta)/\partial\theta=N/\theta-N\bar{y}=0\Rightarrow 1/\theta=\bar{y}\Rightarrow\widehat{\theta}=1/\bar{y}$.

(c) $p(\theta|\mathbf{y})\propto p(\boldsymbol{\theta})\times p(\mathbf{y}|\boldsymbol{\theta})=[e^{-\theta}]\times\theta^N e^{-\theta N\bar{y}}=\theta^N e^{-\theta(N\bar{y}+1)}$.

(d) Part (c) is gamma density $f(\theta)\propto\theta^{a=1}e^{-\theta/b}$ with $a=(N+1)$ and $b=1/(N\bar{y}+1)$.
The posterior mean is $\mathrm{E}[\theta]=ab=(N+1)\times[1/(N\bar{y}+1)]=(N+1)/(N\bar{y}+1)$.

1-2 (a) $p(\mathbf{y}|\theta)=\prod_{i-1}^{N}p(\mathbf{y}_i|\theta)=\prod_{i-1}^{N}(2\pi)^{-1/2}\exp[-(y_i-\theta)^2/2]$
$=(2\pi)^{-N/2}\exp[-\sum_i(y_i-\theta)^2/2]$.
So $p(\mathbf{y}|\theta)\propto\exp[-\sum_i(y_i-\theta)^2/2]=\exp[\{-\sum_i(y_i-\bar{y})^2/2\}-N(\bar{y}-\theta)^2/2]$
(given hint.) $=\exp[-\sum_i(y_i-\bar{y})^2/2]\exp[-N(\bar{y}-\theta)^2/2]$
The first term does not involve θ so $p(\mathbf{y}|\theta)\propto\exp[-N(\bar{y}-\theta)^2/2]$

(b) MLE minimizes ln$p(\mathbf{y}|\theta)\propto\ln\{\exp[N(\bar{y}-\theta)^2/2]\}=N(\bar{y}-\theta)^2/2]$.
$\partial\ln p(\mathbf{y}|\theta)/\partial\theta=-N(\bar{y}-\theta)=0\Rightarrow\widehat{\theta}=\bar{y}$.

(c) $p(\theta|\mathbf{y})\propto p(\boldsymbol{\theta})\times p(\mathbf{y}|\boldsymbol{\theta})\propto[\exp(-\theta^2/2)]\times\exp[-N(\bar{y}-\theta)^2/2]$
$=\exp\{-\frac{1}{2}[N(\bar{y}-\theta)^2+\theta^2]$.

(d) $N(\bar{y}-\theta)^2+\theta^2=N\bar{y}^2-2N\theta\bar{y}+N\theta^2+\theta^2=(N+1)\theta^2-2N\theta\bar{y}+N\bar{y}^2$
$=(N+1)\{\theta^2-2\frac{N}{N+!}\theta\bar{y}\}+N\bar{y}^2=(N+1)\{\theta^2-\frac{N}{N+1}\bar{y}\}^2-(N+1)(\frac{N}{N+1}\bar{y})^2+N\bar{y}^2$.

(e) $p(\theta|\mathbf{y})\propto\exp\{-\frac{1}{2}[N(\bar{y}-\theta)^2+\theta^2]$
$\propto\exp\{-\frac{1}{2}[(N+1)\{\theta-\frac{N}{N+!}\bar{y}\}^2-(N+1)(\frac{N}{N+1}\bar{y})^2+N\bar{y}^2]\}$
$=\exp\{-\frac{1}{2}[(N+1)\{\theta-\frac{N}{N+1}\bar{y}\}^2]\}\times\exp\{-\frac{1}{2}[(N+1)(\frac{N}{N+1}\bar{y})^2+N\bar{y}^2]\}$
$\propto\exp\{-\frac{1}{2}[(N+1)\{\theta-\frac{N}{N+1}\bar{y}\}^2]\}\propto\exp\{-\frac{1}{2\times[1/(N+1)]}\{\theta-\frac{N}{N+!}\bar{y}\}^2]\}$.
This is the kernel of a normal density with mean $\frac{N}{N+1}\bar{y}$ and variance $1/(N+1)$.
The posterior mean is $\frac{N}{N+1}\bar{y}$.

(f) From preceding the posterior variance is $1/(N+1)$.
The MLE is $\bar{y}$ which for $y\sim N[0,1]$ has variance $1/N$ that exceeds $1/(N+1)$.
Intuitively, using prior information improves precision.

1-3 To do.
This question uses simulated methods to calculate moments, using $\widehat{\mathrm{E}}[g(\boldsymbol{\theta})]=\frac{1}{S}\sum_{s=1}^{S}g(\boldsymbol{\theta}^{(s)})$.

Suppose the posterior density for parameter θ is $N[0,1]$.
(a) Calculate the posterior mean of θ using $S = 10$, 100, 1000 and $10,000$ draws.
(b) Calculate the variance of this simulation estimate when $S = 100$ (see Koop).
(c) Calculate $\mathrm{E}[\exp(-\exp(x))]$ using $S = 10$, 100, 1000 and $10,000$ draws.
(d) Does $\mathrm{E}[\exp(-\exp(x))]$ actually exist? [Hint: This is tricky and can be skipped.]