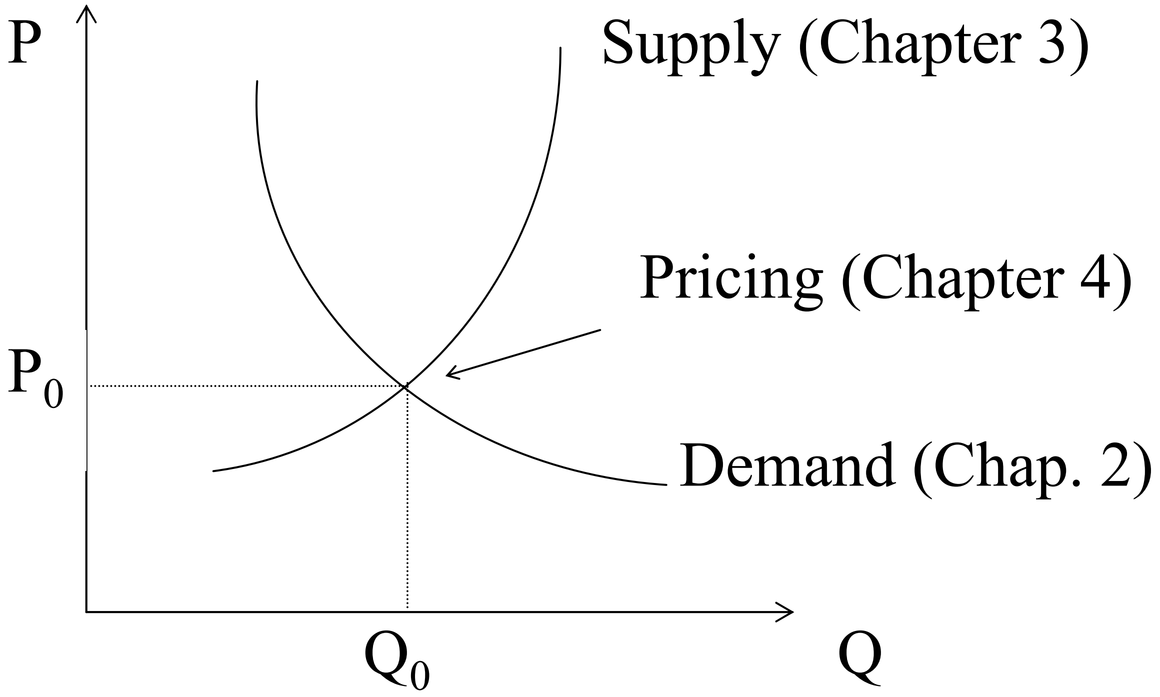
ECN 145 Lecture 12

Transportation Economics: Production and Costs I

Demand and Supply

- Chapters from *Essays* textbook:



How to obtain supply?

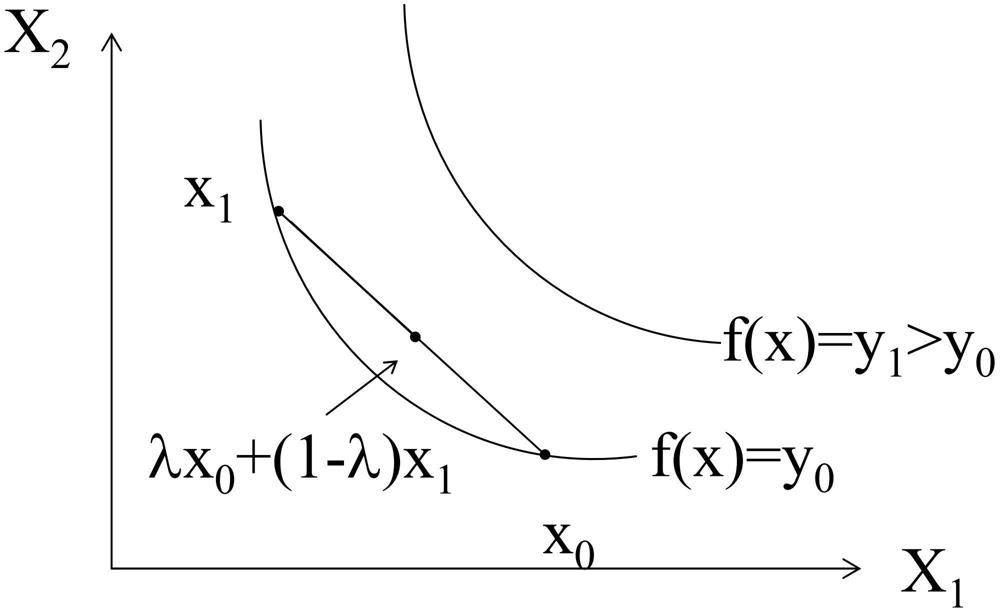
- 1)What is the production technology?
- 2)Therefore, what are production costs?
- 3)Given demand, what will the firm (or government) choose to supply?
- **We will focus here on <u>production and costs</u>.**

Production function:

- Given inputs $x=(x_1,x_2,\ldots,x_n)$, write output y as, $y = f(x_1,\ldots,x_n) = f(x)$
- Properties:
  - **Increasing:** $\partial f / \partial x > 0$
  - **Quasi-concave**:
  - if $f(x_0) = f(x_1)$ and $0 \le \lambda \le 1$
  - then $f(\lambda x_0 + (1-\lambda)x_1) \ge f(x_0)$

Production Function

- Illustrate with “iso-quants”:



Returns to Scale

- Suppose that there is one input x, and,

$$y=f(x)=x^{\alpha}$$

- $\alpha=1 \Rightarrow$ constant returns to scale
- doubling all inputs will just double output
- $\alpha>1 \Rightarrow$ Increasing returns to scale
- doubling all inputs <u>more than</u> doubles output
- $\alpha<1 \Rightarrow$ decreasing returns to scale.
- doubling all inputs will <u>less than</u> double output

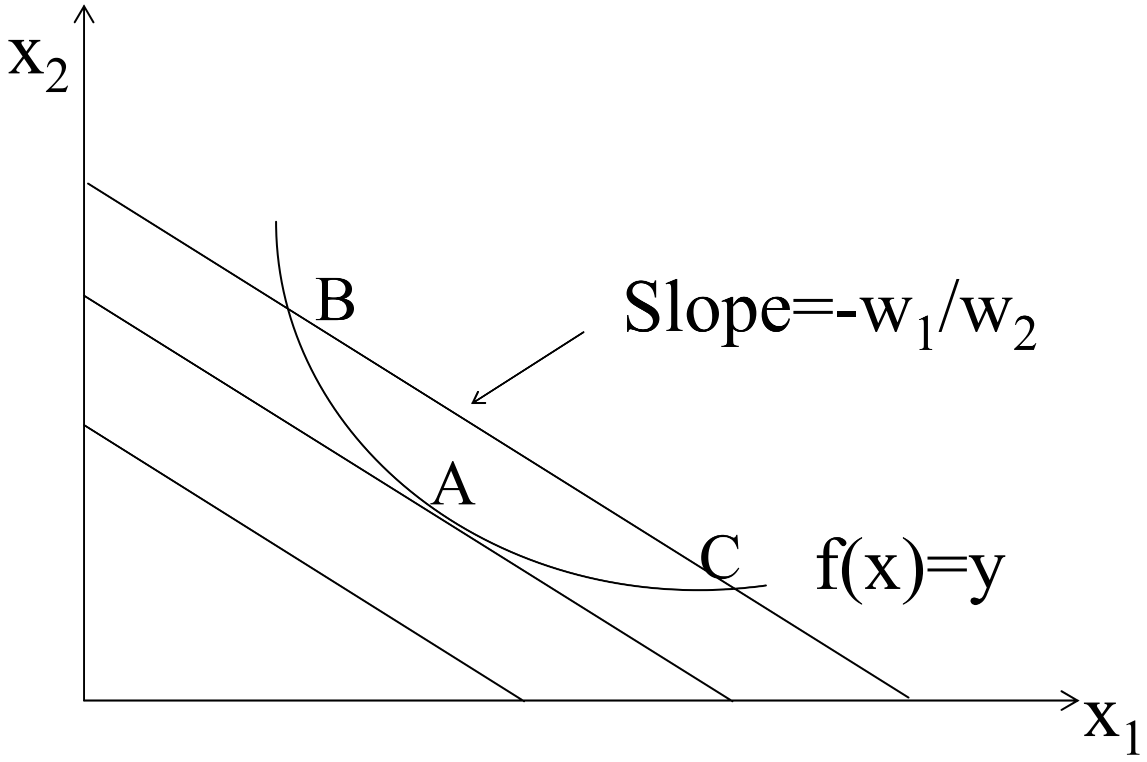
Input prices

- Prices for inputs x_i are w_i
- E.g. labor – wage
- Capital – rental price
- Fuel – cost of oil;
- Total costs are,

$$C = \sum_{i=1}^{n} w_i x_i$$

The firm's problem:

$$\min \sum_{i=1}^{n} w_i x_i \quad \text{subject to } f(x) \geq y$$



The firm’s problem (cont’d):

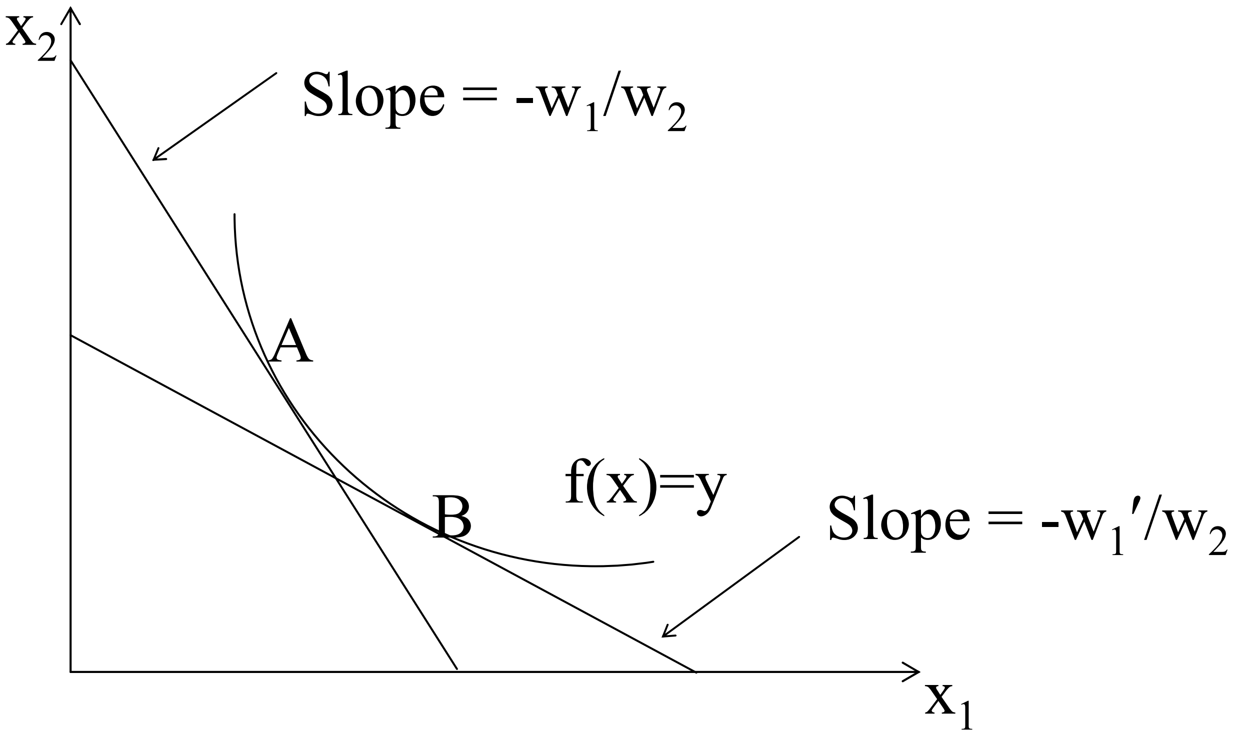
- Point A is the lowest cost method of producing y
  - **B and C are more expensive to get y**
- Write solution as **cost function**:

$$C(y,w)=\sum_{i=1}^{n} w_i x_i(y,w)$$

- with:
- $x_i(y,w)$ - input demands

Change in Price:

- Suppose price falls from w_1 to w_1':

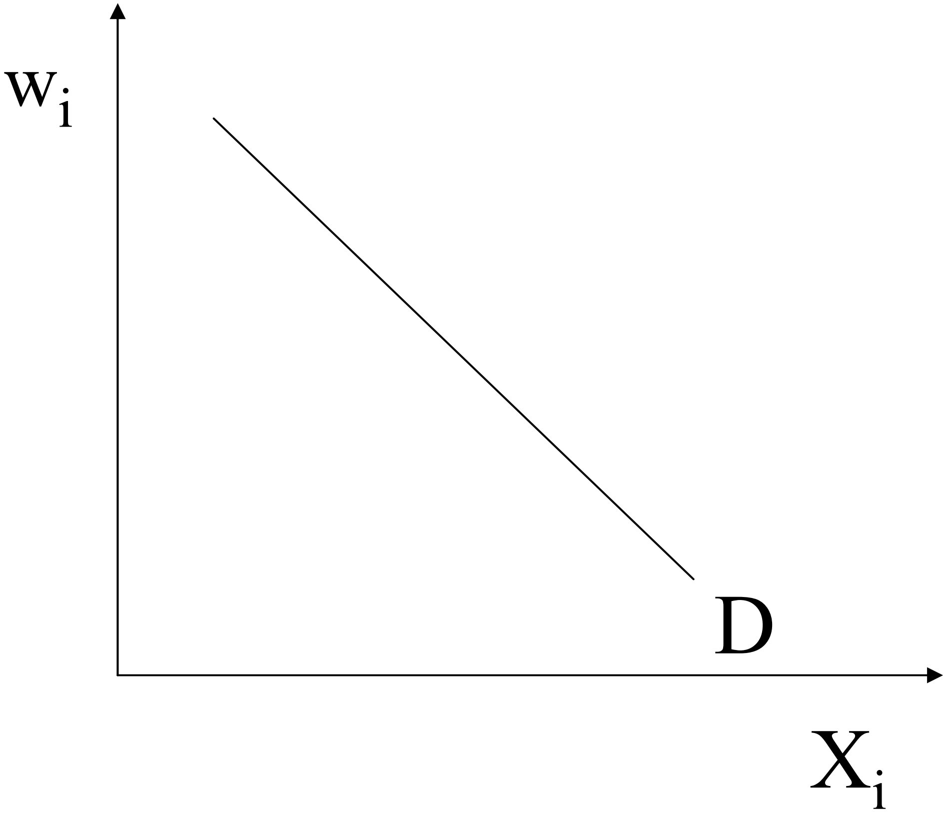


Change in Price (cont'd):

- Fall in w_1 will <u>increase</u> demand for x_1, and reduce demand for x_2, moving from A to B.
- $\frac{\partial x_i}{\partial w_i} < 0$ - pure "substitution" effect

Demand Curve

- This gives us downward sloping demand:



Derivative of costs:

$$\frac{\partial C}{\partial w_i} = x_i(y, w) + \sum_{j=1}^{n} w_j \frac{\partial x_j}{\partial w_i} = x_i(y, w)$$

- Because $\sum_{j=1}^{n} w_j \frac{\partial x_j}{\partial w_i} = 0$
  - **(substitution effects all cancel out)**

- Thus, the derivative of costs w.r.t. factor prices equals *factor demands*

Returns to Scale:

- If, $$y = f(x) = x^{\alpha}$$
- then, $$C(y, w) = y^{1/\alpha} w$$
- $\alpha = 1 \Rightarrow$ doubling output will double costs;
- $\alpha > 1 \Rightarrow$ Increasing returns to scale.
- doubling output will less than double costs;
- $\alpha < 1 \Rightarrow$ Decreasing returns to scale.
- doubling output will more than double costs.

Average costs:

- Hold the (single) input price w fixed:
- Define *average costs*,

$$AC = \frac{C(y, w)}{y} = \frac{y^{1/\alpha} w}{y} = w y^{\frac{1-\alpha}{\alpha}},$$

- E.g. Total costs =$100, y=5, so AC=$20

Marginal costs

- Hold the (single) input price w fixed:
- Define *marginal costs,*

$$MC = \frac{\partial C(y, w)}{\partial y} = \frac{w}{\alpha} y^{\frac{1-\alpha}{\alpha}}$$

- E.g. Total costs=$100 when y=5, $115 when y=6
- So marginal costs are $15.

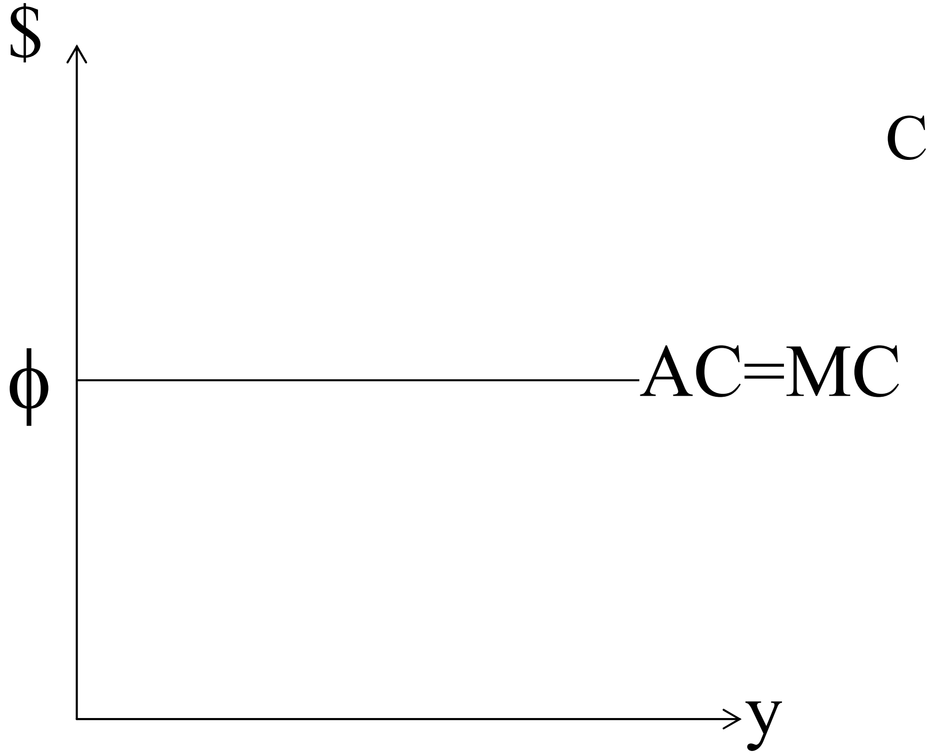
Returns to Scale:

$$AC = wy^{\frac{1-\alpha}{\alpha}}, \qquad MC = \frac{1}{\alpha} wy^{\frac{1-\alpha}{\alpha}}$$

- so, $\dfrac{AC}{MC} = \alpha = \dfrac{\text{Total Costs}}{y \cdot MC}$,
- which is a measure of returns to scale!

Constant Returns to Scale:

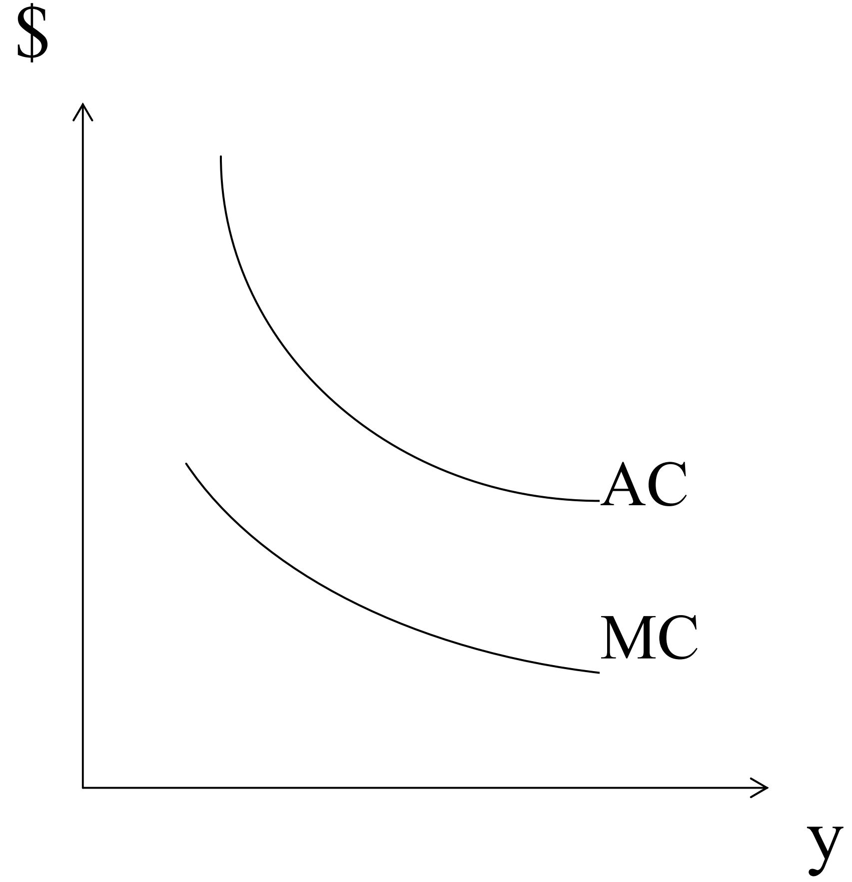
$\alpha=1$



Constant returns

Increasing Returns to Scale:

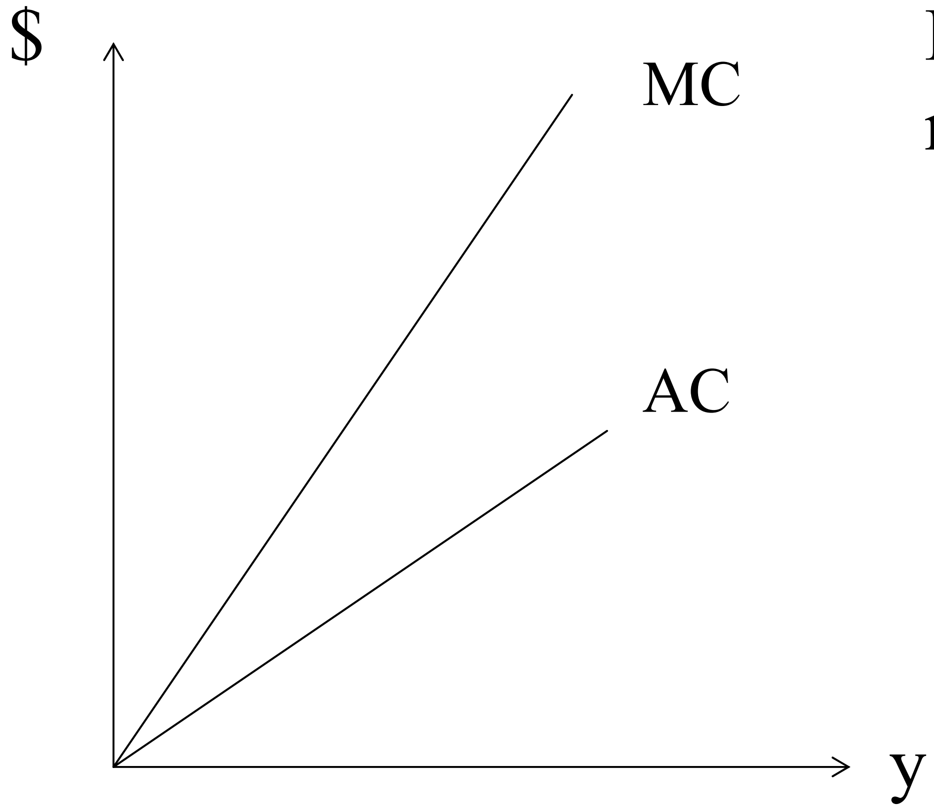
$\alpha > 1$



Increasing returns

Decreasing Returns to Scale:

$\alpha<1$ (Say, $\alpha=1/2$)



Decreasing returns

Eg: Cobb-Douglas Production Function

$$y = f(x) = x_1^{\alpha} x_2^{\beta}, \ \alpha, \beta > 0$$

- Notice that doubling both x_1 and x_2 :

$$f(2x) = (2x_1)^{\alpha}(2x_2)^{\beta} = 2^{\alpha+\beta} f(x)$$

- So,
- $\alpha + \beta = 1 \Rightarrow$ constant returns to scale
- $\alpha + \beta > 1 \Rightarrow$ increasing returns to scale
- $\alpha + \beta < 1 \Rightarrow$ decreasing returns to scale

Cobb-Douglas Cost Function

$$\min w_1 x_1 + w_2 x_2 \text{ subject to } x_1^{\alpha} x_2^{\beta} \geq y$$

- Use a Lagrangian,

$$L = w_1 x_1 + w_2 x_2 + \lambda(y - x_1^{\alpha} x_2^{\beta})$$

- Find first-order condition w.r.t x_1, x_2:

$$\frac{\partial L}{\partial x_1} = w_1 - \lambda \alpha x_1^{\alpha-1} x_2^{\beta} = 0 \Rightarrow w_1 x_1 = \lambda \alpha f(x)$$

$$\frac{\partial L}{\partial x_2} = w_2 - \lambda \beta x_1^{\alpha} x_2^{\beta-1} = 0 \Rightarrow w_2 x_2 = \lambda \beta f(x)$$

Cobb-Douglas Cost Function (cont'd)

- From these two conditions, we solve for,

$$w_1 x_1 + w_2 x_2 = \lambda(\alpha + \beta) f(x)$$

- We can solve for λ from the FOC,

$$w_1^{\alpha} x_1^{\alpha} = \lambda^{\alpha} \alpha^{\alpha} f(x)^{\alpha}, \quad w_2^{\beta} x_2^{\beta} = \lambda^{\beta} \beta^{\beta} f(x)^{\beta}$$

$$\Rightarrow (w_1^{\alpha} w_2^{\beta}) f(x) = \lambda^{\alpha+\beta} (\alpha^{\alpha} \beta^{\beta}) f(x)^{\alpha+\beta}$$

Cobb-Douglas Cost Function (cont'd)

- Solving for λ, costs then are,

$$C(y, w_1, w_2) = w_1 x_1 + w_2 x_2$$

$$= \lambda(\alpha+\beta) f(x) = (\alpha+\beta) A y^{\frac{1}{\alpha+\beta}} (w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}})$$

- where,
- $A = (\alpha^{-\frac{\alpha}{\alpha+\beta}} \beta^{-\frac{\beta}{\alpha+\beta}})$ is a constant.

Returns to Scale:

- First write the log of costs as:

$$\ln C = B + \left(\frac{1}{\alpha+\beta}\right)\ln y + \left(\frac{\alpha}{\alpha+\beta}\right)\ln w_1 + \left(\frac{\beta}{\alpha+\beta}\right)\ln w_2$$

- Differentiating this w.r.t. y, we see that,

$$\frac{AC}{MC} = \frac{\text{Costs}}{y(\partial C/\partial y)} = \left(\frac{\partial \ln C}{\partial \ln y}\right)^{-1} = (\alpha+\beta)$$

- measures the returns to scale!

Cobb-Douglas Input Demands:

- Demands for inputs are obtained by differentiating costs:

$$x_1(y, w_1, w_2) = \frac{\partial C}{\partial w_1} = \alpha A y^{\frac{1}{\alpha+\beta}} w_1^{\frac{\alpha-1}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}}$$

$$x_2(y, w_1, w_2) = \frac{\partial C}{\partial w_2} = \beta A y^{\frac{1}{\alpha+\beta}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta-1}{\alpha+\beta}}$$

Costs Shares:

- Comparing x_1 and x_2 with total costs:

$$C = (\alpha + \beta) A y^{\frac{1}{\alpha+\beta}} \left(w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} \right)$$

- we see that,

$$\frac{w_1 x_1}{C} = \frac{\alpha}{\alpha + \beta}$$

-

$$\frac{w_2 x_2}{C} = \frac{\beta}{\alpha + \beta}$$

which are constant!

Elasticity of Substitution

- For the Cobb-Douglas function:

$$\frac{x_1}{x_2} = \frac{\alpha}{\beta}\frac{w_2}{w_1}$$

- So the elasticity of substitution is,

$$-\frac{\partial \ln(x_1 / x_2)}{\partial \ln(w_1 / w_2)} = 1$$

- This may not be a good description of actual substitution between inputs! So consider…..

Translog Cost Function

- Many outputs (joint prod.) $y = (y_1, \cdots, y_m)$
- And inputs $w = (w_1, \cdots, w_n)$

$$\ln C(y, w) = a_0 + \sum_{i=1}^{m} a_i \ln y_i + \sum_{i=1}^{n} b_i \ln w_i$$

$$+ \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \ln y_i \ln y_j + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \ln w_i \ln w_j$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij} \ln y_i \ln w_j$$

Translog Cost Function (cont'd)

- Note that the first line is just Cobb-Douglas,
- (in logs, with multiple inputs and outputs):

$$\ln C(y, w) = a_0 + \sum_{i=1}^{m} a_i \ln y_i + \sum_{i=1}^{n} b_i \ln w_i$$

- The "extra" terms on the second and third lines allow for more general substitution between inputs and outputs.

Translog Cost Shares:

- Differentiating the cost function,

$$\frac{\partial \ln C}{\partial \ln w_i} = \frac{x_i w_i}{C} = b_i + \sum_{j=1}^{n} b_{ij} \ln w_j + \sum_{i=1}^{m} g_{ij} \ln y_j$$

- so that,

$$\frac{\partial^2 \ln C}{\partial \ln w_i \partial \ln w_j} = b_{ij} \neq 0$$

- this allows for a wide pattern of substitution between inputs.

Returns to Scale:

$$\frac{AC}{MC} = \frac{\text{Costs}}{y \cdot (\partial C / \partial y)} = \left(\sum_i \partial \ln C / \partial \ln y_i \right)^{-1}$$

- so if a_{ij}=0, then, $\frac{AC}{MC} = \left(\sum_i a_i \right)^{-1}$
- is a measure of returns to scale!