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MIDTERM EXAM (total 100 points)

1. [40 points] Consider a duopoly with differentiated goods and price competition. For simplicity, assume that the firms have **zero costs**. The demand function of firm $i \in \{1,2\}$ is given by

$D_i = 15A_i - 2p_i + p_j \quad (j \neq i)$ where $A_i \geq 1$ is a parameter that can be interpreted as measuring the quality of the product of firm *i*. Consider two scenarios.

Scenario 1: the firms choose their prices simultaneously and independently.

(a) [15 points] Find the Nash equilibrium of the game of Scenario 1.

Scenario 2: The game is played in three stages: in Stage 1 Firm 1 chooses the value of $A_1 \in [1,\infty)$; in Stage 2 the chosen value of A_1 has become common knowledge between the two firms and now Firm 2 chooses the value of $A_2 \in [1,\infty)$; in Stage 3 the chosen values of A_1 and A_2 are common knowledge between the two firms and now the two firms choose their prices simultaneously and independently.

(b) [25 points] Find the subgame-perfect equilibrium of the game of Scenario 2 assuming that the choice of $A_i \in [1,\infty)$ is costly and the cost is different for the two firms:

$$c_1(A_1) = 10(A_1)^3 + 20(A_1)^2 - 30 \quad \text{and} \quad c_2(A_2) = 40\left[(A_2)^2 - 1\right]$$

2. [50 points] Consider a homogeneous duopoly with inverse demand $P = 240 - 2Q$ and cost functions

$$C_1(q_1) = \begin{cases} 0 & \text{if } q_1 = 0 \\ 4q_1 + 450 & \text{if } q_1 > 0 \end{cases} \text{ and } C_2(q_2) = \begin{cases} 0 & \text{if } q_2 = 0 \\ 8q_2 + 450 & \text{if } q_2 > 0 \end{cases}.$$

(a) [10 points] Suppose first that the two firms play a simultaneous game. Find the Cournot-Nash equilibrium and calculate the profit of each firm at the equilibrium.

(b) [35 points] Suppose now that Firm 1 moves first and commits to an output level. Firm 2 observes Firm 1's commitment and chooses its own output level (thus it is a Stackelberg game). Assume that Firm 2 chooses a positive level of output if and only if it expects to make positive profits. Find the backward-induction solution.

(c) [5 points] Are consumers better off if the two firms play the game of Part (a) or if they play the game of Part (b)?

3. [10 points] Consider the following two-player game:
$S_1 = S_2 = [0,1]$, $\pi_1 : [0,1]^2 \to \mathbb{R}$ and $\pi_2 : [0,1]^2 \to \mathbb{R}$ are as follows:

$$\pi_1(x_1, x_2) = \begin{cases} x_1 & \text{if } (x_1, x_2) \neq (1,1) \\ 0 & \text{if } (x_1, x_2) = (1,1) \end{cases} \quad \text{and} \quad \pi_2(x_1, x_2) = \begin{cases} x_2 & \text{if } (x_1, x_2) \neq (1,1) \\ 0 & \text{if } (x_1, x_2) = (1,1) \end{cases}$$

Prove that this game has (1) no pure-strategy Nash equilibria and (2) not mixed-strategy Nash equilibria if the support of each mixed strategy is finite.

1. **(a)** The profit function of firm *i* is $\Pi_i = p_i D_i$. The solution to $\frac{\partial \Pi_1}{\partial p_1} = 0, \frac{\partial \Pi_2}{\partial p_2} = 0$ is

$$p_1 = 4A_1 + A_2, \quad p_2 = A_1 + 4A_2$$

(b) The Nash equilibrium of the last stage is given in part (a). The corresponding profits at the Nash equilibrium are:

$$\Pi_1(A_1, A_2) = 2\left(4A_1 + A_2\right)^2 \quad \text{and} \quad \Pi_2(A_1, A_2) = 2\left(A_1 + 4A_2\right)^2$$

In Stage 2, firm 2 chooses A_2 to maximize

$$f_2(A_1, A_2) = \Pi_2(A_1, A_2) - 40\left(A_2^2 - 1\right) = 2\left(A_1 + 4A_2\right)^2 - 40\left(A_2^2 - 1\right)$$

The solution to $\frac{\partial f_2}{\partial A_2} = 0$ is $A_2(A_1) = A_1$. Thus in Stage 1 firm 1 chooses A_1 to maximize

$$f_1(A_1) = \Pi_1(A_1, A_1) - 10A_1^3 - 20A_1^2 + 30 = 25A_1^2 - 10A_1^3 - 20A_1^2 + 30$$

The solution to $\frac{df_1}{dA_1} = 0$ is $A_1 = 2$ (note that $\frac{d^2 f_1}{dA_1^2}(2) = -60$ thus it is indeed a maximum).

Hence the subgame-perfect equilibrium is as follows:

Strategy of firm 1: $\left(A_1 = 2,\ p_1(A_1, A_2) = 4A_1 + A_2\right)$,

Strategy of firm 2: $\left(A_2(A_1) = A_1,\ p_2(A_1, A_2) = A_1 + 4A_2\right)$.

Note that it would be wrong to state that the subgame-perfect equilibrium is $A_1 = A_2 = 2$ and $p_1 = p_2 = 10$, because subgame-perfect equilibria are defined in term of strategy profiles and a strategy of a player has to cover every possible contingency, not only the one that arises if the equilibrium is played.

2. **(a)** The profit functions are as follows:

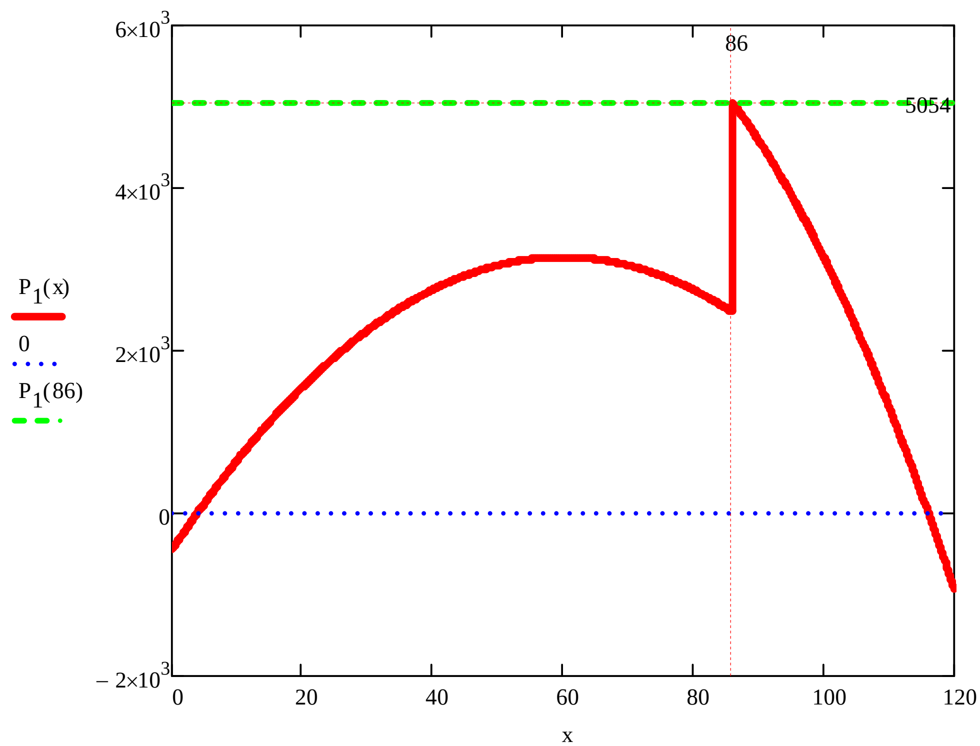
$$\pi_1(q_1, q_2) = q_1\left[240 - 2(q_1 + q_2)\right] - 4q_1 - 450$$
$$\pi_2(q_1, q_2) = q_2\left[240 - 2(q_1 + q_2)\right] - 8q_2 - 450$$

The CNE is found by solving $\begin{cases} \dfrac{\partial \pi_1}{\partial q_1} = 0 \\ \dfrac{\partial \pi_2}{\partial q_2} = 0 \end{cases}$. The solution is $q_1^* = 40,\ q_2^* = 38$ with corresponding price and profits of $\boxed{P = 84}$, $\pi_1^* = 2{,}750,\ \ \pi_2^* = 2{,}438$.

(b) We need to construct Firm 2's best reply function. First we solve $\dfrac{\partial \pi_2}{\partial q_2} = 0$ to get $q_2 = 58 - \dfrac{q_1}{2}$. Then we solve $\pi_2\left(q_1,\ 58 - \frac{q_1}{2}\right) = 0$ to get $q_1 = 86$. Thus Firm 2's best reply function is as follows:

$$BR_2(q_1) = \begin{cases} 0 & \text{if } q_1 \geq 86 \\ 58 - \dfrac{q_1}{2} & \text{if } q_1 < 86 \end{cases}$$

Hence Firm 1 chooses q_1 to maximize $\pi_1\left(q_1, BR_2(q_1)\right)$. The graph of this function is shown below:



Thus the backward induction solution is $\left(86,\ BR_2(q_1)\right)$. The outcome is $q_1 = 86,\ q_2 = 0$ with corresponding price and profits of $\boxed{P = 68}$, $\pi_1 = 5{,}054,\ \ \pi_2 = 0$.

(c) Since the price is lower at the Stackelberg equilibrium than at the Cournot equilibrium, consumers are better off at the former.

3.

(1) Suppose that player 1's strategy is $x_1 = 1$. Then player 2 has no best reply, because if she chooses $x_2 = 1$ her payoff is 0 whereas if she chooses $x_2 < 1$ her payoff is x_2 (thus she would want to choose the largest x_2 which is strictly less than 1 and, of course, there is no such number because of the open interval problem).
Suppose that player 1's pure strategy is $\hat{x}_1 < 1$. Then player 2's best response is $x_2 = 1$ but then, by the above argument, $\hat{x}_1$ is not a best reply to $x_2 = 1$.

(2) If player 1 chooses a mixed strategy with finite support that assigns zero probability to $x_1 = 1$ (that is, $\sigma_1(1) = 0$) then 2's best reply is $x_2 = 1$ but then (by point (1) above) player 1's mixed strategy is not a best reply to $x_2 = 1$. Suppose that player 1 chooses a mixed strategy σ_1 with finite support that assigns positive probability to $x_1 = 1$ (that is, $\sigma_1(1) = \alpha$ with $0 < \alpha < 1$: the case where $\sigma_1(1) = 1$ was considered in point (1)). Then player 2 has no best reply, because if she chooses $x_2 = 1$ then her payoff is $0\alpha + 1(1-\alpha) = 1-\alpha$ whereas if she chooses x_2 arbitrarily close to 1 then her payoff is x_2 (thus she would want to choose the largest x_2 which is strictly less than 1 and, of course, there is no such number because of the open interval problem).

University of California, Davis - Department of Economics

SPRING 2021 **ECN / ARE 200C: MICROECONOMIC THEORY** Professor Giacomo Bonanno

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FINAL EXAM

ANSWER ALL QUESTIONS (total 100 points)

1. [40 points] There are two types of individuals: Type *L* and Type *H*. They all derive utility from *m* (money) and *y* (education). The utility functions are:

$$\text{for Type L:} \quad U_L(m,y) = 50 + 5m - 2y^2$$
$$\text{for Type H:} \quad U_H(m,y) = 50 + 5m - y^2$$

Education affects productivity (denoted by π) as follows.

$$\text{for Type L:} \quad \pi_L(y) = 4 + y$$
$$\text{for Type H:} \quad \pi_H(y) = 4 + \tfrac{5}{4}y$$

(a) [8 points] Suppose first that employers are able tell the two types apart and offer to pay each individual an amount of money *m* equal to the person's productivity (a job applicant's level of education is observed by the employer). For each type calculate the amount of education that the individual chooses, the amount of money she is offered and her level of utility.

(b) From now on assume that employers **cannot** tell whether job applicants are of Type L or Type H. However, they can observe their level of education. Suppose that employers offer to pay new hires according to the following rule: if your level of education is *y* then I will pay you *w*(*y*) where $w(y) = \begin{cases} 4 + y & \text{if } y < y^* \\ 4 + \frac{5}{4}y & \text{if } y \geq y^* \end{cases}$. For each of the following cases either calculate a separating equilibrium (and prove that it is indeed an equilibrium) or show that a separating equilibrium does not exist. *As part of the definition of separating equilibrium we require that each worker is paid an amount that is equal to her actual productivity*.

(b.1) [8 points] $y^* = 2$;

(b.2) [8 points] $y^* = 3$;

(b.3) [8 points] $y^* = 4.5$;

(b.4) [8 points] $y^* = 5.5$.

2. [60 points] There is a single seller of a used car and a single buyer, whose initial wealth is \$4,891 and utility-of-money function is $U(\$x) = \sqrt{x}$. The buyer believes that the seller's car is of low quality (L) with probability 60% and of high quality H with probability 40%. The buyer considers owning a car of quality L to be equivalent to an increase in her wealth by \$109 and owning a car of quality H to be equivalent to an increase in her wealth by \$400.

The seller can get a written and accurate assessment of the quality of the car by an independent third party at zero cost. The seller's utility is:

- if his car is of quality $H: \begin{cases} 22.5 & \text{if he does not sell the car} \\ \frac{1}{10}(P-25) & \text{if he sells the car at price } P \end{cases}$

- if his car is of quality $L: \begin{cases} 7.5 & \text{if he does not sell the car} \\ \frac{1}{10}(P-25) & \text{if he sells the car at price } P \end{cases}$

All of the above is common knowledge among the two players.

Consider the following game: first the seller decides whether or not to obtain an independent assessment, then the buyer observes whether or not the written assessment was obtained and, if it was, she learns its content and then offers a price P to the seller, which the seller then either accepts or rejects. Assume that, if indifferent between accepting and rejecting, the seller will accept and that is common knowledge between buyer and seller.

(a) [15 points] Sketch the extensive-form game that is obtained by applying the Harsanyi transformation to this situation of incomplete information.

(b) [20 points] Find all the separating pure-strategy weak sequential equilibria and calculate the expected payoffs of both players at each equilibrium.

(c) [20 points]Consider all the candidates for pooling pure-strategy weak sequential equilibria and for each such candidate either prove that it is a weak sequential equilibrium, and calculate the expected payoffs of both players, or prove that it is not a weak sequential equilibrium.

(d) [5 points] Should the seller get the assessment?

1. **(a)** **Type L** chooses y to maximize $U_L(m,y)=50+5m-2y^2$ subject to $m=4+y$, that is, chooses y to maximize $70+5y-2y^2$. The solution is ***y* = 1.25** with m = 5.25 and $U_L=73.125$.

Type H chooses y to maximize $U_H(m,y)=50+5m-y^2$ subject to $m=4+\frac{5}{4}y$, that is, chooses y to maximize $70+4y-y^2$. The solution is ***y* = 3.125** with m = 7.91 and $U_H=79.77$.

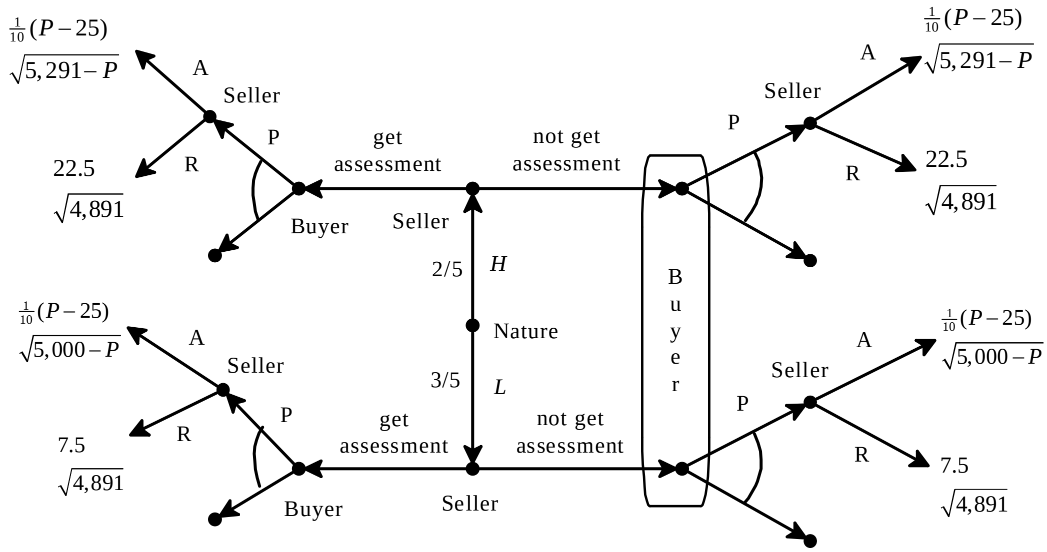
(b) **(b.1)** In this case ($y^*=2$) there is no separating equilibrium. At such an equilibrium, in order for Type L individuals to be paid according to their true productivity, they would have to choose $y<y^*$. From part (a) we know that the highest utility a Type L person can get subject to $y<y^*$ is obtained when y = 1.25 and is given by $U_L=73.125$. On the other hand, if a Type L chose y = 2, then she would be paid 4 + 1.25(2) = 6.5 and her utility would be $U_L=74.5$. Thus she would prefer "masquerading as a Type H person" by choosing a y of at least 2 and would therefore not be paid according to her true productivity (e.g. if y = 2 her true productivity is 6 rather than 6.5).

(b.2) In this case ($y^*=3$) we have a separating equilibrium where Type L workers choose y = 1.25, are paid m = 5.25 and their utility is $U_L=73.125$ and Type H workers choose y = 3.125, are paid m = 7.91 and their utility is $U_H=79.77$. Proof that this is an equilibrium. We know from part (a) that for Type H, y = 3.125 maximizes $U_H(m,y)=50+5m-y^2$ subject to $m=4+\frac{5}{4}y$ for every y (thus in particular for $y \geq 3$); on the other hand, for $y<y^*=3$, $U_H(m,y)=50+5m-y^2$ with $m=4+y$ is maximized at y = 2.5 with corresponding utility of 76.25 for type H which is less than the utility of 79.77 that they get with y = 3.125. For a Type L, y = 1.25 maximizes $U_L(m,y)=50+5m-2y^2$ subject to $m=4+y$, for every y (thus in particular for y < 3); if a Type L person chose $y \geq 3$, then her utility would be $U_L(4+\frac{5}{4}y,y)=70+6.25y-2y^2$, which is decreasing in y in the range $t \geq 3$; thus the highest value in this range is t = 3 with a utility of 70.75, which is less than the utility of choosing y = 1.25 (which is 73.125).

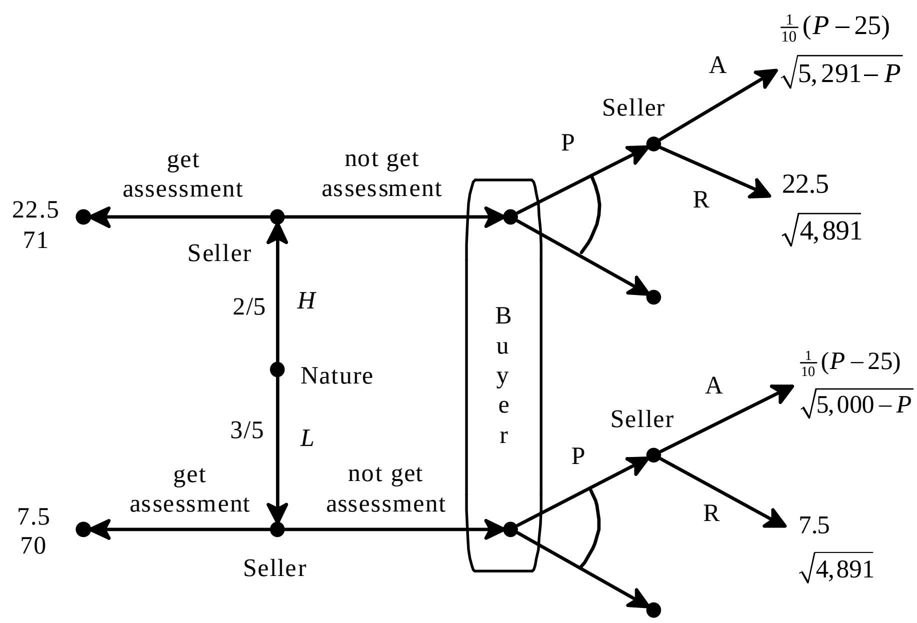
(b.3) In this case ($y^*=4.5$) we have a separating equilibrium where Type L workers choose y = 1.25, are paid m = 5.25 and their utility is $U_L=73.125$ and Type H workers choose y = 4.5, are paid m = 9.625 and their utility is $U_H=77.875$. Proof that this is an equilibrium. For Type L workers the proof is the same as in part (b.2). For Type H, $U_H(m,y)=50+5m-y^2$ subject to $m=4+\frac{5}{4}y$ is decreasing in the range $y \geq 4.5$, so that the optimal value of y in this range is y = 4.5 with a corresponding utility of 77.875; on the other hand, for $y<4.5$, $U_H(m,y)=50+5m-y^2$ with $m=4+y$ is maximized at y = 2.5 with a utility of 76.25.

(b.4) In this case ($y^*=5.5$) there is no separating equilibrium. By the same argument as in part (b.3), for a Type H the optimal value of y in the range $y \geq 5.5$ is y = 5.5 with a corresponding utility of 74.125; thus a Type H would prefer "masquerading as a Type L person" by choosing y = 2.5 (as shown in part (b.3)) obtaining a utility of 76.25 and would therefore not be paid according to her true productivity.

2. **(a)** The game is as follows.



(b) If the quality is H the seller will accept to sell if and only if $P \geq 250$ and if the quality is L the seller will accept to sell if and only if $P \geq 100$. Thus these are the prices that the buyer will offer if she learns the quality of the car from the written assessment. Note that $\sqrt{4,891} = 69.94$, $\sqrt{5,291-250} = 71$ and $\sqrt{5,000-100} = 70$. Thus the game can be simplified as follows:



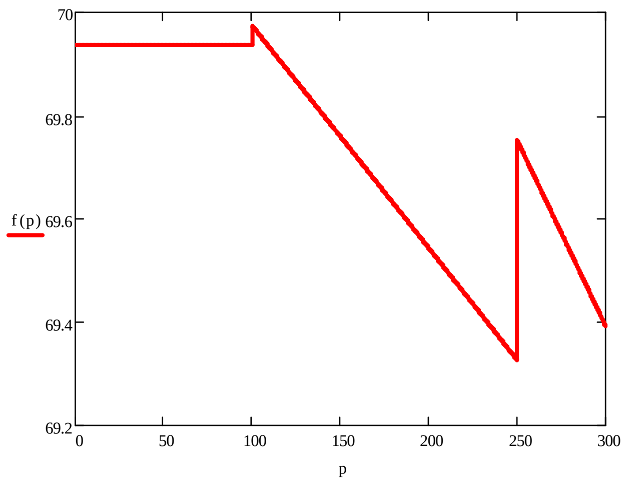
There is only one pure-strategy separating weak sequential equilibrium: the seller's strategy is to have the car assessed if it is of quality H and not assessed if it is of quality L (and then, after any offer, accept any $P \geq 250$ if H and any $P \geq 100$ if L). In this case the buyer must assign probability 1 to the lower node of her information set and then she will offer a price of \$100 which the seller will accept. Thus the buyer's strategy is: offer \$250 at top-left node, offer \$100 at bottom left node, offer \$100 at information set on the right. The seller's expected payoff is $\frac{2}{5}22.5 + \frac{3}{5}7.5 = 13.5$ and the buyer's expected payoff is $\frac{2}{5}71 + \frac{3}{5}70 = 70.4$

The other candidate for a pure-strategy weak sequential equilibrium would be the following. Seller's strategy: have the car assessed if it is of quality *L* and not assessed if it is of quality *H* (and then, after any offer, accept any $P \geq 250$ if *H* and any $P \geq 100$ if *L*). In this case the buyer must assign probability 1 to the top node of her information set and then she will offer a price of \$250 which the seller will accept. But then it would not be rational for the L-type of the seller to get the assessment, because with the assessment his utility is 7.5, while by switching to no assessment he would be offered P = 250 and his utility would be 22.5.

(c) Let us first consider the seller's pooling strategy of **not** getting an assessment. The buyers' beliefs at the information on set on the right are then her initial beliefs. Let *P* be the price offered at this information set; then the buyer's expected payoff at this information set is

$$f(P) = \begin{cases} \sqrt{4{,}891} & \text{if } P < 100 \\ \frac{2}{5}\sqrt{4{,}891} + \frac{3}{5}\sqrt{5{,}000 - P} & \text{if } 100 \leq P < 250 \\ \frac{2}{5}\sqrt{5{,}291 - P} + \frac{3}{5}\sqrt{5{,}000 - P} & \text{if } P \geq 250 \end{cases}$$

whose graph is shown below:



It is maximized at *P* = 100. Thus at the top node the seller is indifferent between getting and not getting an assessment (his payoff is 22.5 in either case) and the same is true at the bottom node of the seller (his payoff is 7.5 in either case). Hence this is indeed a weak sequential equilibrium with expected payoffs of $\frac{2}{5}22.5 + \frac{3}{5}7.5 = 13.5$ for the seller and $f(100) = \frac{2}{5}\sqrt{4{,}891} + \frac{3}{5}\sqrt{4{,}900} = 69.97$ for the buyer.

The other candidate for a pooling equilibrium has the seller getting an assessment for both quality levels. In this case, the buyer will react to the assessment by offering to pay \$250 if she learns that the quality is *H* and by offering to pay \$100 if she learns that the quality is *L*. We need to complete the buyer's strategy by specifying what she would do at the information set on the right. There we can take any beliefs we like, for example probability 1 at the bottom node, so that she would offer to pay \$100. The seller then would be indifferent between getting and not getting an assessment at both of his nodes. The expected payoffs are then $\frac{2}{5}22.5 + \frac{3}{5}7.5 = 13.5$ for the seller and $\frac{2}{5}71 + \frac{3}{5}70 = 70.4$ for the buyer.

(d) Based on the above analysis, the seller gets a higher expected payoff if either he gets an assessment for both qualities or if he gets an assessment for only the high quality.