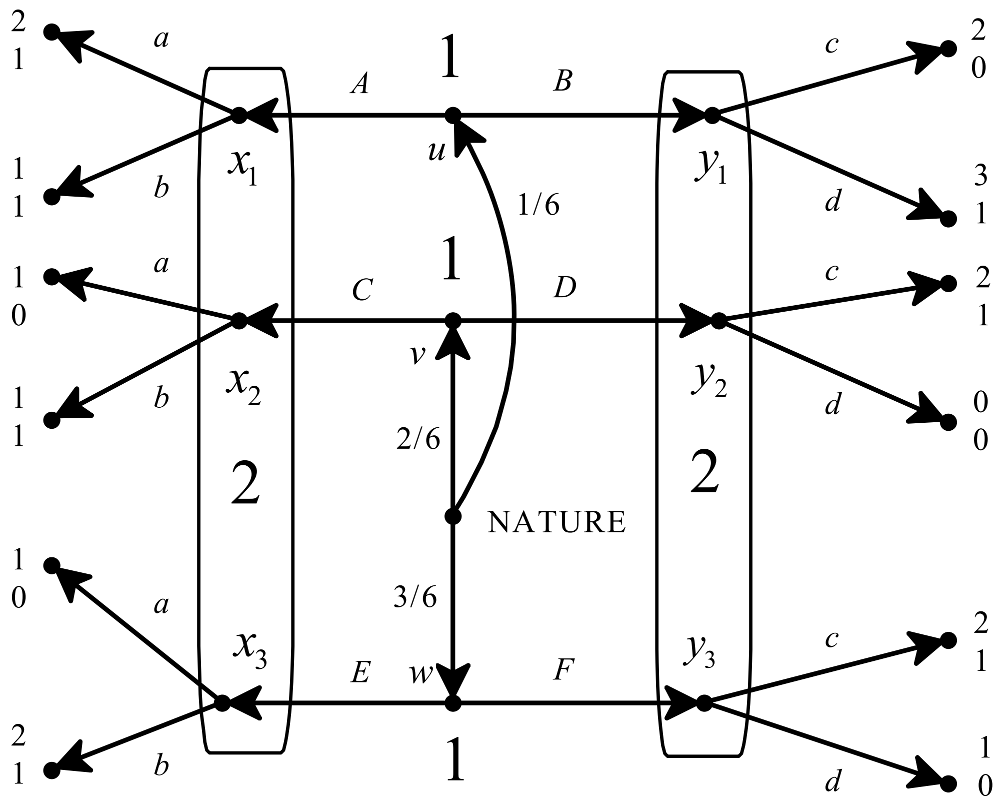
**MIDTERM EXAM**

ANSWER ALL QUESTIONS (total 100 points)

**1.** [12 points] Consider the following game. An individual (not to be thought of as a player) auctions a gold coin worth \$6,535 to $n$ players ($n > 1$). Each player independently submits an envelope containing his/her bid in cash (any amount of cash can be put in the envelope and submitting an empty envelope is allowed). If one player's bid exceeds all the other bids, then he/she wins the coin. If two or more players bid the highest amount, the auctioneer keeps the coin, that is, nobody wins. The submitted envelopes are never returned to the players, that is, no player ever recovers his/her bid. Assume that all the players are selfish and greedy, that is, each player cares only about his/her own wealth and prefers more money to less.

Does this game have any pure-strategy Nash equilibria? If Yes, find at least one. If No, prove it.

**2.** [40 points] Consider the following game where the payoffs are von Neumann-Morgenstern payoffs (in each pair the top number is Player 1's payoff and the bottom number Player 2's payoff).




**(a)** [36 points] Find **all** the pure-strategy weak sequential equilibria.

**(b)** [4 points] Is there a completely mixed-strategy weak sequential equilibrium, that is, a weak sequential equilibrium where each choice is selected with positive probability?

3. [48 points] Consider a market with two firms, both of which have **zero production costs**. We will consider two scenarios, one involving price competition and the other involving competition in output levels.

SCENARIO 1 The two firms play the following two-stage game. In stage 1 they simultaneously and independently choose between H and L, where H means "produce a high-quality product" and L means "produce a low-quality product". At the end of the first stage the two decisions become common knowledge and we proceed to the second stage. In the second stage the firms simultaneously and independently choose the price of their product. Payoffs are given by profits (= revenue). If both firms choose H in stage 1 then they are producing a homogeneous product for which industry demand is $Q = 80 - 8P$, while if they both choose L in stage 1 then they are producing a homogeneous product for which industry demand is $Q = 80 - 10P$. If one chooses H and the other L, then (letting $p_H$ be the price charged by the firm that chose H and $p_L$ the price charged by the firm that chose L) demand is as follows: for the H-firm $q_H = 80 - 40p_H + 40p_L$ and for the L firm $q_L = 40p_H - 50p_L$.

**(a)** [8 points] Sketch the extensive-form game for the case where only two prices are possible: p and p′ (don't worry about payoffs, only about the structure of the game).

**(b)** [6 points] Write down one strategy of firm 1 in the game of part (a) above. How many strategies does firm 1 have in the game of part (a) above?

*From now on allow for all non-negative prices*, that is $p_1 \in [0, \infty)$ and $p_2 \in [0, \infty)$.

**(c)** [4 points] Write the demand function of firm 1, as a function of the two prices $p_1$ and $p_2$, for the case where they both choose H in stage 1. [Recall that the products are identical.]

**(d)** [10 points] Find the pure-strategy subgame-perfect equilibria of the two-stage game.

SCENARIO 2 The two firms play the following two-stage game. The first stage is as in Scenario 1. In the second stage the firms simultaneously and independently choose their output levels. Demand for the case where both firms choose H and the case where both choose L is as given in Scenario 1. For the case where one chooses H and the other chooses L, inverting the demand system given under Scenario 1 yields the following: $p_H = 10 - \frac{q_H}{8} - \frac{q_L}{10}$ and $p_L = 8 - \frac{q_H}{10} - \frac{q_L}{10}$.

**(e)** [16 points] Find the pure-strategy subgame-perfect equilibria of the two-stage game.

**(f)** [4 points] What is the main difference in the firms' behavior when we switch the second-stage game from price competition to output competition?

SPRING 2019 **ECON. 200C: MICROECONOMIC THEORY** Giacomo Bonanno

Midterm Exam ANSWERS

**1.** Recall that $n > 1$ and that a player can win only if he bids a positive amount. Let $(x_1,...,x_n)$ be the bids. Let $x_i$ be a highest bid, that is, $x_i \geq x$ for all $x \in \{x_1,...,x_n\}$.

- If $x_i > 0$, it cannot be that $x_i = x_j$ for some $j \neq i$ (nobody wins and i and j have to pay a positive amount). Thus $x_j < x_i$ for all $j \neq i$; if there is a $j \neq i$ such that $0 < x_j < x_i$ then it is not a Nash equilibrium (player j would do better by reducing his bid to zero). Thus it must be $x_j = 0$ for every $j \neq i$. But then in order for this to be a Nash equilibrium $x_i$ must be the smallest possible amount, that is 1 cent (otherwise player i can increase his payoff by reducing his bid). But the situation where $x_i$ is equal to 1 cent and $x_j = 0$ for every $j \neq i$ is not a Nash equilibrium because a player $j \neq i$ can increase his payoff by bidding 2 cents.
- If, on the other hand, $x_i = 0$, then any player can increase his payoff by bidding 1 cent.

Thus there are no pure-strategy Nash equilibria.

**2.**

**(a)** First note that, at 2's information set on the left, *a* is weakly dominated by *b* and *a* is sequentially rational only if Player 2 assigns probability 1 to node $x_1$, which requires Player 1's pure strategy to be either (*A,D,F*) or (*B,D,F*). In order for *A* to be rational for Player 1 (when Player 2 plays *a* at the left information set) it is necessary and sufficient that Player 2's choice be *c* at her right information set (restricting attention to pure strategies). When Player 2 plays *c* at her right information set and *a* at her left information set then both *D* and *F* are sequentially rational for Player 1. Finally, when Player 1's strategy is (*A,D,F*) then Player 2 assigns probability 0 to node $y_1$ and thus *c* is indeed sequentially rational. Hence the following is a weak sequential equilibrium:

$$\sigma = \big((A,D,F),(a,c)\big),\ \ \mu = \left(\begin{array}{ccc|ccc} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ 1 & 0 & 0 & 0 & \frac{2}{5} & \frac{3}{5} \end{array}\right).$$

A second weak sequential equilibrium is $\sigma = \big((B,D,F),(a,c)\big),\ \ \mu = \left(\begin{array}{ccc|ccc} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ 1 & 0 & 0 & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} \end{array}\right)$

Sequential rationality for Player 1 is satisfied. For Player 2 at the left information set *a* is sequentially rational, given her beliefs and at the right information set her beliefs must be $\begin{pmatrix} y_1 & y_2 & y_3 \\ \frac{1}{6} & \frac{2}{6} & \frac{3}{6} \end{pmatrix}$ yielding the following expected payoffs:

$\pi_2(c) = \frac{1}{6}0 + \frac{2}{6}1 + \frac{3}{6}1 = \frac{5}{6} > \pi_2(d) = \frac{1}{6}1 + \frac{2}{6}0 + \frac{3}{6}0 = \frac{1}{6}$.

There is no other weak sequential equilibrium where Player 2 plays *a* at her left information set. Thus let us consider possible equilibria where Player 2 plays *b*. Then for Player 1, at node *u*, *B* is strictly better than $A$ (no matter what Player 2 does at her right information set). If Player 2 plays $c$ at her right information set, then – for Player 1 – $D$ is better than $C$ and $F$ is better than $E$. Thus we are considering the strategy profile $\sigma = ((B,D,F),(b,c))$. Sequential rationality for Player 1 is satisfied. For Player 2 at the left information set any beliefs make $b$ sequentially rational and at the right information set her beliefs must be $\begin{pmatrix} y_1 & y_2 & y_3 \\ \frac{1}{6} & \frac{2}{6} & \frac{3}{6} \end{pmatrix}$ making $c$ sequentially rational, as shown above. Hence the following is a weak sequential equilibrium, for every $p$ and $q$:

$$\sigma = ((B,D,F),(b,c)), \quad \mu = \left(\begin{array}{ccc|ccc} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ p & q & 1-p-q & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} \end{array}\right).$$

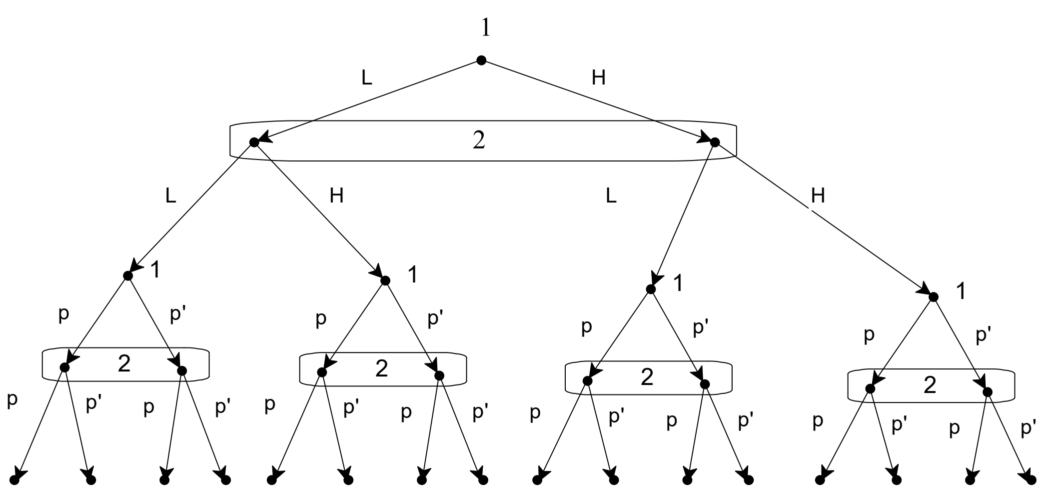
Another weak sequential equilibrium is $\sigma = ((B,D,E),(b,c))$, $\mu = \left(\begin{array}{ccc|ccc} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{2}{3} & 0 \end{array}\right)$.

Finally, we must consider the possibility that Player 2's strategy is $(b,d)$. Since $d$ is rational only if Player 2 assigns sufficiently high probability to node $y_1$, Player 1's strategy must be $(B,C,E)$: each of these choices is rational for Player 1 if Player 2's strategy is $(b,d)$. Thus the third, and last, weak sequential equilibrium is:

$$\sigma = ((B,C,E),(b,d)), \quad \mu = \left(\begin{array}{ccc|ccc} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ 0 & \frac{2}{5} & \frac{3}{5} & 1 & 0 & 0 \end{array}\right)$$

**(b)** No, because Player 2 would have to be indifferent between $a$ and $b$ at her left information set (in order to randomize there) and that happens only if the probability of $x_1$ is 1, which – by Bayes' rule – cannot be the case if Player 1 chooses $C$ and $E$ with positive probability.

**3.** **(a)** The extensive form is as follows:




**(b)** (L, p, p′, p′, p) (going from left to right). Firm 1 has $2^5 = 32$ strategies.

**(c)** $D_1(p_1,p_2)=\begin{cases} 80-8p_1 & if\ \ p_1<p_2 \\ \frac{1}{2}\left(80-8p_1\right) & if\ \ p_1=p_2 \\ 0 & if\ \ p_1>p_2 \end{cases}$

**(d)** In the subgame where they both choose H as well as in the subgame where they both choose L, by Bertrand's theorem the unique Nash equilibrium is $p_1=p_2=0$ with corresponding profits of zero for both firm. Now consider a subgame where one firm chooses H and the other chooses L. The profit functions are $\pi_H=p_H\left(80-40p_H+40p_L\right)$ and $\pi_L=p_L\left(40p_H-50p_L\right)$. To find the Nash equilibrium solve $\frac{\partial\pi_H}{\partial p_H}=0$ and $\frac{\partial\pi_L}{\partial p_L}=0$. The solution is $\left(p_H=\frac{5}{4}=1.25, p_L=\frac{1}{2}=0.5\right)$ with corresponding profits $\pi_H=\frac{125}{2}=62.5$ and $\pi_L=\frac{25}{2}=12.5$. Thus the game can be reduced to the following one-stage game:

| | | Firm 2 | |
|---|---|---|---|
| | | H | L |
| Firm 1 | H | 0 , 0 | 62.5 , 12.5 |
| | L | 12.5 , 62.5 | 0 , 0 |

Thus there are two subgame-perfect equilibria:

$$\left(\underbrace{(H,\text{if HH}\ p_1=0,\ \text{if HL}\ p_1=1.25,\ \text{if LH}\ p_1=0.5,\ \text{if LL}\ p_1=0)}_{\text{firm 1's strategy}},\underbrace{(L,\text{if HH}\ p_2=0,\ \text{if HL}\ p_2=0.5,\ \text{if LH}\ p_2=1.25,\ \text{if LL}\ p_2=0)}_{\text{firm 2's strategy}}\right)$$

where firm 1 chooses H and sets a price of 1.25 and firm 2 chooses L and sets a price of 0.5, and

$$\left(\underbrace{(L,\text{if HH}\ p_1=0,\ \text{if HL}\ p_1=1.25,\ \text{if LH}\ p_1=0.5,\ \text{if LL}\ p_1=0)}_{\text{firm 1's strategy}},\underbrace{(H,\text{if HH}\ p_2=0,\ \text{if HL}\ p_2=0.5,\ \text{if LH}\ p_2=1.25,\ \text{if LL}\ p_2=0)}_{\text{firm 2's strategy}}\right)$$

where firm 1 chooses L and sets a price of 0.5 and firm 2 chooses H and sets a price of 1.25.

**(e)** In the subgame where they both choose H, inverse demand is $P=10-\frac{Q}{8}$ so that the profit functions are $\pi_1=q_1\left(10-\frac{q_1+q_2}{8}\right)$ and $\pi_2=q_2\left(10-\frac{q_1+q_2}{8}\right)$. To find the Nash equilibrium solve $\frac{\partial\pi_1}{\partial q_1}=0$ and $\frac{\partial\pi_2}{\partial q_2}=0$. The solution is $q_1=q_2=\frac{80}{3}=26.67$ with corresponding profits $\pi_1=\pi_2=\frac{800}{9}=88.89$.

In the subgame where they both choose L, inverse demand is $P=8-\frac{Q}{10}$ so that the profit functions are $\pi_1=q_1\left(8-\frac{q_1+q_2}{10}\right)$ and $\pi_2=q_2\left(8-\frac{q_1+q_2}{10}\right)$. To find the Nash equilibrium solve

$\frac{\partial \pi_1}{\partial q_1} = 0$ and $\frac{\partial \pi_2}{\partial q_2} = 0$. The solution is $q_1 = q_2 = \frac{80}{3} = 26.67$ with corresponding profits $\pi_1 = \pi_2 = \frac{640}{9} = 71.11$

Now consider a subgame where one firm chooses H and the other chooses L. The profit functions are $\pi_H = q_H \left( 10 - \frac{q_H}{8} - \frac{q_L}{10} \right)$ and $\pi_L = q_L \left( 8 - \frac{q_H}{10} - \frac{q_L}{10} \right)$. To find the Nash equilibrium solve $\frac{\partial \pi_H}{\partial q_H} = 0$ and $\frac{\partial \pi_L}{\partial q_L} = 0$. The solution is $\left( q_H = 30, q_L = 25 \right)$ with corresponding profits $\pi_H = \frac{225}{2} = 112.5$ and $\pi_L = \frac{125}{2} = 62.5$. Thus the game can be reduced to the following one-stage game:

| | | Firm 2 | |
|---|---|---|---|
| | | H | L |
| Firm 1 | H | 88.89 , 88.89 | 112.5 , 62.5 |
| | L | 62.5 , 112.5 | 71.11 , 71.11 |

Thus there is a unique subgame-perfect equilibrium where they both choose H and produce 26.67 units each:

strategy of firm 1: $(H, \text{if HH } q_1 = 26.67,\ \text{if HL } q_1 = 30,\ \text{if LH } q_1 = 25,\ \text{if LL } q_1 = 26.67)$

strategy of firm 2: $(H, \text{if HH } q_2 = 26.67,\ \text{if HL } q_2 = 25,\ \text{if LH } q_2 = 30,\ \text{if LL } q_2 = 26.67)$

**(f)** The main difference is that in scenario 1 the firms choose to differentiate their products, while in scenario 2 they choose to produce a homogeneous product.

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**FINAL EXAM**

ANSWER ALL QUESTIONS (total 100 points)

**1.** [33 points] Imagine a world where a person's productivity is decided at birth and education has no effect on it. However, employers don't know this and believe that education is the determinant of productivity. There are two types of individuals. One type is born with a productivity of \$20,000 and has the following cost of acquiring education ($y$ denotes the number of years of schooling): $C_L(y) = a(y-6)$. The other type is born with a productivity of \$27,000 and has the following cost of acquiring education $C_H(y) = b(y-6)$. The possible choices of y are 6, 12, 16, 18 and 21. Employers offer the following wage schedule (erroneously believing that education affects productivity):

| $y$ | wage |
|---|---|
| 6 | \$8,000 |
| 12 | \$20,000 |
| 16 | \$24,000 |
| 18 | \$27,000 |
| 21 | \$30,300 |

**(a)** [24 points] Find **all** the values of $a$ and $b$ that give rise to a signaling equilibrium. [Assume that – when indifferent – an individual chooses the level of education that gives her a salary equal to her true productivity.]

**(b)** [9 points] Assume that $a$ and $b$ are such that a signaling equilibrium exists and, currently, the economy is at a signaling equilibrium. Suppose that employers are risk-neutral. Suppose also that the population is composed as follows: 40% with productivity 20,000 and 60% with productivity 27,000. Would it be desirable (that is, would it lead to a Pareto improvement) to (1) make it compulsory for everybody to complete 12$^{th}$ grade and (2) shut down all the institutions of higher education (that is, any type of education beyond 12th grade)? [If you think that the answer is a function of the values of $a$ and $b$ then be explicit about it; if not state it clearly.]

**2.** [33 points] A monopolist faces $n_A$ type $A$ consumers, $n_B$ type B consumers and $n_C$ type C consumers. Each type $A$ consumer has the demand function $D_A(P) = A - P$, each type $B$ consumer has the demand function $D_B(P) = B - P$ and each type $C$ consumer has the demand function $D_C(P) = C - P$, with $A > B > C > k$ where $k$ is the constant unit cost of production.

**(a)** [6 points] Write the profit maximization problem for the monopolist when it uses bundling rather than linear pricing. Make sure that you write all the relevant constraints.

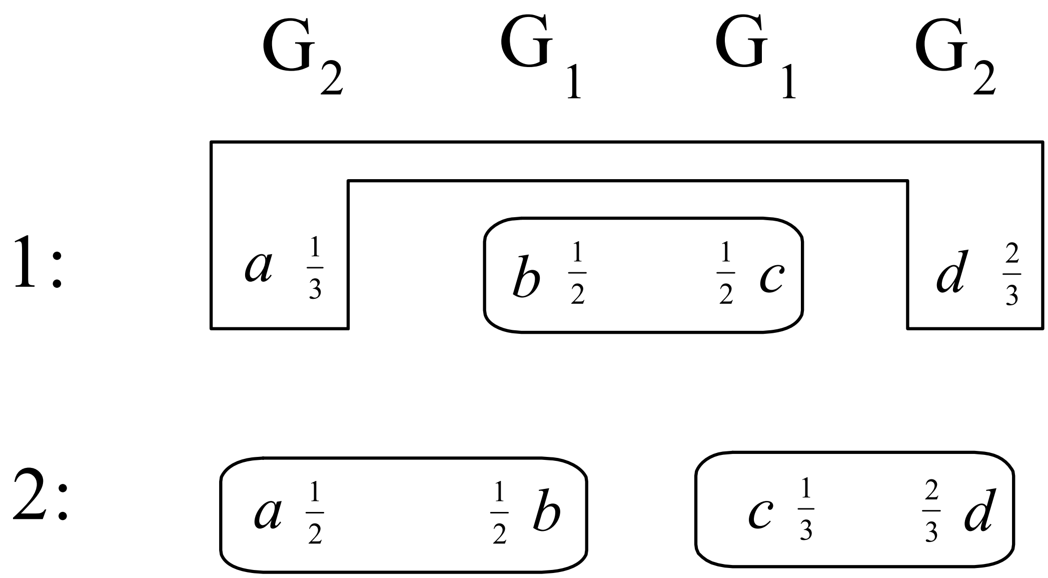
**(b)** [5 points] Show that some of the constraints are redundant.

From now on focus on the case where $\boxed{n_A = n_B = n_C = 1 \text{ and } A = 10,\ B = 8,\ C = 6,\ k = 2}$.

Furthermore, assume that (1) if a consumer is indifferent between two (or more) packages then she purchases the one with the larger (largest) quantity, (2) no consumer is allowed to purchase more than one package.

- **(c)** [9 points] Suppose that the monopolist offers the following three packages (in each pair **the first number is the price of the package and the second number the quantity**): (16,4), (22,6) and (28,8). Calculate the monopolist's profits.
- **(d)** [9 points] Suppose that the monopolist offers only two packages: (30,6) and (36,8). What will its profits be?
- **(e)** [4 points] Can you infer from Parts (c) and (d) that the monopolist should not serve all three consumers? [A brief explanation is sufficient.]

**3.** [34 points] Consider the following situation of two-sided incomplete information:




Where $G_1$ and $G_2$ are the following simultaneous games:

$G_1$

| Player 1 \ Player 2 | $L$ | $R$ |
|---|---|---|
| $T$ | 1 , 0 | 0 , 2 |
| $B$ | 0 , 4 | 2 , 1 |

$G_2$

| Player 1 \ Player 2 | $L$ | $R$ |
|---|---|---|
| $T$ | 0 0 | 3 2 |
| $B$ | 2 4 | 0 1 |

- **(a)** [4 points] The beliefs of the two players are Harsanyi consistent. Find the common prior.
- **(b)** [15 points] Apply the Harsanyi transformation to transform the above situation of incomplete information into an extensive form game. [Make Player 1 move before Player 2.]
- **(c)** [15 points] Is there a weak sequential equilibrium which is completely mixed (that is, every choice is made with positive probability)? If Yes, find it, if No explain why not.

**1.** Recall that – when indifferent – an individual chooses the level of education that gives her a salary equal to her true productivity. Hence inequalities will be taken to be weak inequalities.

**(a)** Decision problem for person of productivity 20,000

| y | gross wage | cost | net income |
|---|---|---|---|
| 6 | 8,000 | 0 | 8,000 |
| 12 | 20,000 | *6a* | *20,000-6a* |
| 16 | 24,000 | *10a* | *24,000-10a* |
| 18 | 27,000 | *12a* | *27,000-12a* |
| 21 | 30,300 | *15a* | *30,300-15a* |

At a signaling equilibrium the best choice must be $y$ = 12 (so that they get paid their true productivity). Hence we need all of the following inequalities to be satisfied:

$$\begin{pmatrix} 20,000-6a \geq 8,000 & \text{that is} & a \leq 2,000 \\ 20,000-6a \geq 24,000-10a & \text{that is} & a \geq 1,000 \\ 20,000-6a \geq 27,000-12a & \text{that is} & a \geq 1,166.66 \\ 20,000-6a \geq 30,300-15a & \text{that is} & a \geq 1,144.44 \end{pmatrix}$$

Thus we need $\boxed{\frac{7,000}{6} = 1,166.66 \leq a \leq 2,000}$

Decision problem for person of productivity 27,000

| y | gross wage | cost | net income |
|---|---|---|---|
| 6 | 8,000 | 0 | 8,000 |
| 12 | 20,000 | *6b* | *20,000-6b* |
| 16 | 24,000 | *10b* | *24,000-10b* |
| 18 | 27,000 | *12b* | *27,000-12b* |
| 21 | 30,300 | *15b* | *30,300-15b* |

At a signaling equilibrium the best choice must be $y$ = 18 (so that they get paid their true productivity). Hence we need all of the following inequalities to be satisfied:

$$\begin{pmatrix} 27,000-12b \geq 8,000 & \text{that is} & b \leq 1,583.33 \\ 27,000-12b \geq 20,000-6b & \text{that is} & b \leq 1,166.66 \\ 27,000-12b \geq 24,000-10b & \text{that is} & b \leq 1,500 \\ 27,000-12b \geq 30,300-15b & \text{that is} & b \geq 1,100 \end{pmatrix}$$

Thus we need $\boxed{1,100 \leq b \leq 1,166.66 = \frac{7,000}{6}}$

**(b)** If schools beyond 12th grade were eliminated, the employers would no longer have education as a signal of productivity. Hiring an employee would then be the same as playing the lottery $\begin{pmatrix} \textit{productivity} & 20,000 & 27,000 \\ \hline \textit{probability} & 0.4 & 0.6 \end{pmatrix}$ which has an expected value of 24,200. Thus employers (being risk-neutral) would pay everybody \$24,200. Employees of type $L$ would be better off (their salaries would increase from 20,000 $-6a$ to 24,200 $-6a$). Employees of type $H$ would be better off if and only if $24,200-6b>27,000-12b$, that is, if and only if $b\geq\frac{2,800}{6}=466.66$. Since a signaling equilibrium requires $b$ to be larger than this, the proposed policy would lead to a Pareto improvement (all employees better off, employers as well off) and this is true for all the values of $a$ and $b$ that are consistent with a signaling equilibrium.

**2.** **(a)** Let $W_i(Q)=\int_0^Q(i-x)\,dx=iQ-\frac{Q^2}{2}$ be the willingness to pay for $Q$ units for a consumer of type $i\in\{A,B,C\}$. Since $A>B>C$,

$$W_A(Q)>W_B(Q)>W_C(Q) \text{ for every } Q>0. \qquad (*)$$

The monopolist offers a menu of packages $\{(V_A,Q_A),(V_B,Q_B),(V_C,Q_C)\}$ to maximize

$$\pi=n_AV_A+n_BV_B+n_CV_C-k(n_AQ_A+n_BQ_B+n_CQ_C)$$

subject to:

(1) Individual rationality constraints $(IR_i)$ : $W_i(Q_i)-V_i\geq 0 \quad (i\in\{A,B,C\})$

(2) Incentive compatibility for type $A$:
$(IC_{AB})\ W_A(Q_A)-V_A\geq W_A(Q_B)-V_B$ and $(IC_{AC})\ W_A(Q_A)-V_A\geq W_A(Q_C)-V_C$

(3) Incentive compatibility for type $B$:
$(IC_{BA})\ W_B(Q_B)-V_B\geq W_B(Q_A)-V_A$ and $(IC_{BC})\ W_B(Q_B)-V_B\geq W_B(Q_C)-V_C$

(4) Incentive compatibility for type $C$:
$(IC_{CA})\ W_C(Q_C)-V_C\geq W_C(Q_A)-V_A$ and $(IC_{CB})\ W_C(Q_C)-V_C\geq W_C(Q_B)-V_B$

**(b)** $IR_A$ and $IR_B$ are redundant: $IR_A$ follows from ($IC_{AC}$), $IR_C$ and (*) and $IR_B$ follows from ($IC_{BC}$), $IR_C$ and (*).

**(c)** The "$C$" consumer gets zero surplus from the package (16,4) and negative surplus from the other two packages. Thus she purchases the (16,4) package.
The "$B$" consumer gets a surplus of 8 from the package (16,4) and also from the package (22,6) and a surplus of 4 from the package (25,8). Thus she purchases the (22,6) package.
The "$A$" consumer gets a surplus of 16 from the package (16,4) and a surplus of 20 from the other two packages. Thus she purchases the (28,8) package. Hence the monopolist's profits are: 28 + 22 + 16 − 2(4 + 6 + 8) = 30.

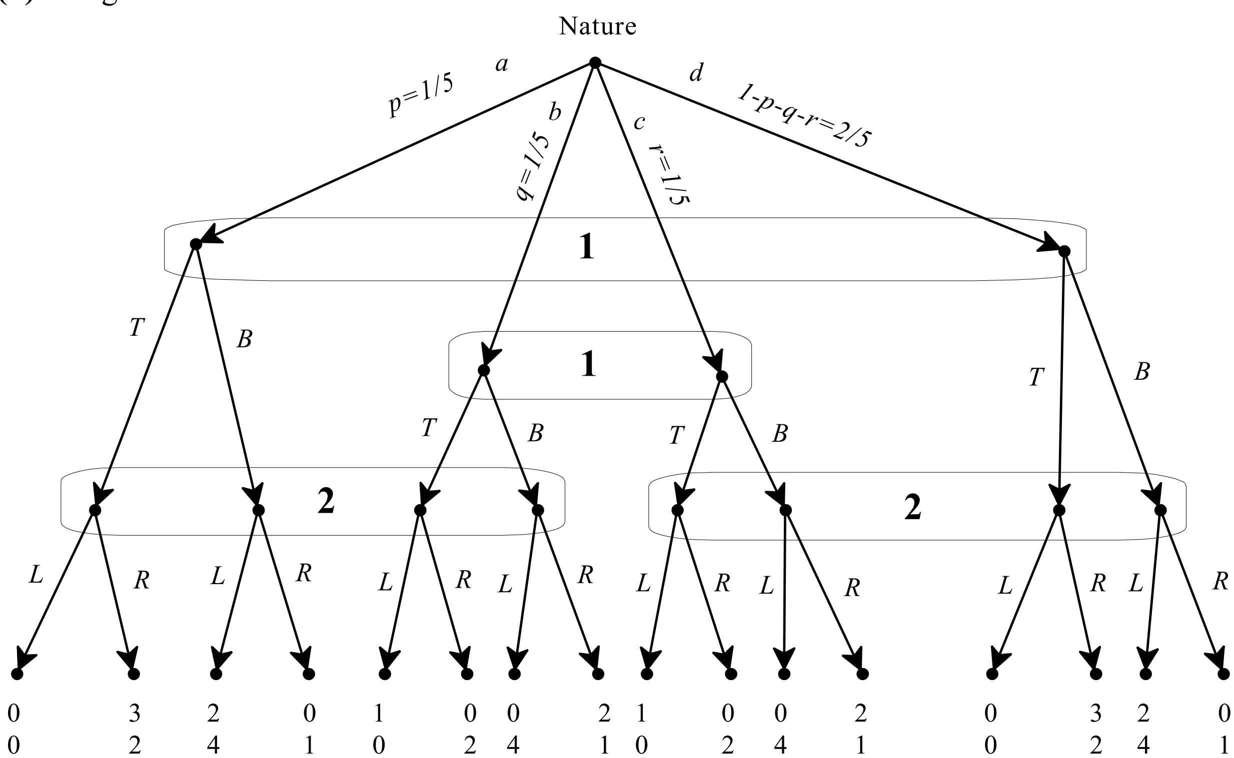
**(d)** The "$C$" consumer gets negative surplus from each package and thus does not buy.
The "$B$" consumer gets zero surplus from the package (30,6) and negative from the other package. Thus she purchases the (30,6) package.
The "$A$" consumer gets a surplus of 12 from both packages and thus she purchases the (36,8) package. Hence the monopolist's profits are: 30 + 36 − 2( 6 + 8) = 38.

**(e)** No, because the fact that the three packages of Part (c) satisfy the constraints does not imply that those three packages are the solution to the maximization problem of Part (a).

**3. (a)** Let $\begin{pmatrix} a & b & c & d \\ p & q & r & 1-p-q-r \end{pmatrix}$ be the common prior. Then it must satisfy the following equations:

$\dfrac{p}{p+1-p-q-r} = \dfrac{1}{3}$, $\dfrac{q}{q+r} = \dfrac{1}{2}$, $\dfrac{p}{p+q} = \dfrac{1}{2}$ and $\dfrac{r}{r+1-p-q-r} = \dfrac{1}{3}$. The solution is $\begin{pmatrix} a & b & c & d \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$.

**(b)** The game is as follows:




**(c)** • First of all, by Bayesian updating the beliefs of Player 1 at his top information set must be 1/3 on the left node and 2/3 on the right node and his beliefs at the lower information set must be ½ on each node.

• In order for Player 1 to be willing to mix at his top information set, he must be getting the same expected payoff from $T$ and from $B$. Let $p$ be the probability with which Player 2 chooses $L$ at her left information set and $q$ the probability with which Player 2 chooses $L$ at her right information set. Then at his top information set Player 1 gets

from $T$: $\frac{1}{3}3(1-p)+\frac{2}{3}3(1-q)=3-p-2q$

from $B$: $\frac{1}{3}2p+\frac{2}{3}2q=\frac{2}{3}(p+2q)$

thus we need $3-p-2q=\frac{2}{3}(p+2q)$, that is, $\boxed{p+2q=\frac{9}{5}}$.

• In order for Player 1 to be willing to mix at his lower information set, he must be getting the same expected payoff from $T$ and from $B$.

from $T$: $\frac{1}{2}p+\frac{1}{2}q$

from $B$: $\frac{1}{2}2(1-p)+\frac{1}{2}2(1-q)=2-p-q$

thus we need $\frac{1}{2}p+\frac{1}{2}q=2-p-q$, that is, $\boxed{p+q=\frac{4}{3}}$.

The solution to these two equations is $\boxed{p=\frac{13}{15},\ q=\frac{7}{15}}$.

• Let $r$ be the probability with which Player 1 plays $T$ at his top information set and $s$ the probability of $T$ at his lower information set. Then, by Bayesian updating, Player 2's beliefs at her left information set must be (from left to right): $\begin{pmatrix} \frac{r}{2} & \frac{1-r}{2} & \frac{s}{2} & \frac{1-s}{2} \end{pmatrix}$ and her beliefs at her right information set must be (from left to right): $\begin{pmatrix} \frac{s}{3} & \frac{1-s}{3} & \frac{2r}{3} & \frac{2(1-r)}{3} \end{pmatrix}$.

• In order for Player 2 to be willing to mix at her **left** information set, she must be getting the same expected payoff from $L$ and from $R$.

from $L$: $\frac{1-r}{2}4 + \frac{1-s}{2}4 = 2(2-r-s)$

from $R$: $\frac{r}{2}2 + \frac{1-r}{2} + \frac{s}{2}2 + \frac{1-s}{2} = \frac{1}{2}(2+r+s)$

thus we need $2(2-r-s) = \frac{1}{2}(2+r+s)$, that is, $\boxed{r+s=\frac{6}{5}}$.

• In order for Player 2 to be willing to mix at her **right** information set, she must be getting the same expected payoff from $L$ and from $R$.

from $L$: $\frac{1-s}{3}4 + \frac{2(1-r)}{3}4 = \frac{4}{3}(3-2r-s)$

from $R$: $\frac{s}{3}2 + \frac{1-s}{3} + \frac{2r}{3}2 + \frac{2(1-r)}{3} = \frac{1}{3}(3+2r+s)$

thus we need $\frac{4}{3}(3-2r-s) = \frac{1}{3}(3+2r+s)$, that is, $\boxed{2r+s=\frac{9}{5}}$.

The solution to these two equations is $\boxed{r=\frac{3}{5},\ s=\frac{3}{5}}$.

Hence the completely mixed strategy weak sequential equilibrium is as follows:

Player 1's behavior strategy at both of his information sets: $\begin{pmatrix} T & B \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$

Player 2's behavior strategy at her left information set: $\begin{pmatrix} L & R \\ \frac{13}{15} & \frac{2}{15} \end{pmatrix}$

Player 2's behavior strategy at her right information set: $\begin{pmatrix} L & R \\ \frac{7}{15} & \frac{8}{15} \end{pmatrix}$

Player 1's beliefs at his top information set: 1/3 on the left node and 2/3 on the right node, and his beliefs at the lower information set: ½ on each node.

Player 2's beliefs at her left information (from left to right): $\begin{pmatrix} \frac{3}{10} & \frac{2}{10} & \frac{3}{10} & \frac{2}{10} \end{pmatrix}$ and her beliefs at her right information set (from left to right): $\begin{pmatrix} \frac{3}{15} & \frac{2}{15} & \frac{6}{15} & \frac{4}{15} \end{pmatrix}$.