## MIDTERM EXAM

ANSWER ALL QUESTIONS (total 100 points)

1. [25 points] A seller has $m$ identical units $(m \geq 1)$ of a commodity for sale. There are $n$ bidders $(n>m)$, each wishing to purchase at most one unit. All the bidders have the same valuation for one unit of the good, given by $\$ V>0$. The seller asks each bidder $i$ to submit a sealed bid $b_{i}$, which can be any non-negative real number. The bids are submitted simultaneously. The seller then arranges the bids in decreasing order (from highest to lowest) and then sells the $m$ units to the first $m$ bidders on that list and charges each of them the $(m+1)^{\text {th }}$ bid. If there are ties, then the seller favors the bidder(s) with the lower index (e.g. bidder 5 has a lower index than bidder 8). For example, suppose that $m=5, n=7$ and the bids are as follows: $b_{1}=\$ 3, b_{2}=\$ 6, b_{3}=\$ 7, b_{4}=\$ 3, b_{5}=\$ 5, b_{6}=\$ 2, b_{7}=\$ 3$. Then the seller arranges the bids as follows: $b_{3}=7, b_{2}=6, b_{5}=5, b_{1}=3, b_{4}=3, b_{7}=3, b_{6}=2$ and sells 1 unit to each of bidders $3,2,5,1$ and 4 for $\$ 3$ (the sixth bid on that list).
(a) [ 3 points] What is this auction called if $m=1$ ?
(b) $[12$ points $]$ For the case where $m>1$, show that each player has a weakly dominant strategy.
(c) [7 points] Show that the strategy of part (b) is not strictly dominant.
(d) [3 points] What is the seller's revenue at the dominant-strategy equilibrium?
2. [27 points] Player 1 (he) is sitting in front of a computer terminal and is about to play a game. Player 2 (she) is sitting in a separate room. Player 1 is told that, when he clicks on START, the computer will randomly pick whether the opponent is another computer (this happens with probability $r, \quad 0<r<1$ ) or Player 2 (this happens with probability $1-r$ ). Player 1 is not informed of what the computer selected. Player 1 then has to choose one of two options: $A$ or $B$. In the case where the opponent is a computer, the computer-opponent receives Player 1's choice as input and is programmed to make the same choice as Player 1. In the case where the opponent is Player 2, she is informed that she has to play and she also has to choose between $A$ and $B$, but she is not informed of the choice made by Player 1. The von Neumann-Morgenstern payoffs are as follows, with $b>c>0$ :

## Player 2

Player 1
B


If the opponent is a computer then Player 1 gets the payoffs corresponding to $(A, A)$ or $(B, B)$ in the above table and Player 2 gets a payoff of 0 . All of the above is common knowledge between Player 1 and Player 2.
(a) [8 points] Draw an extensive-form game to represent the situation described above.
(b) [5 points] Write the corresponding strategic form.
(c) [3 points] For what values of $r$ does Player 1 have a strictly dominant strategy?
(d) [3 points] For what values of $r$ does Player 2 have a strictly dominant strategy?
(e) [8 points] Find all the (pure- and mixed-strategy) Nash equilibria for all the possible values of $r$.
3. [28 points]. Until last year the sunscreen lotion industry consisted of only one firm, call it firm 1. An entrepreneur, Ms Moove, has decided to enter the industry. She expects that, after entry of her new firm (call it firm 2), there will be a Cournot duopoly: the two firms will independently choose their output levels and the price will be determined by the inverse demand function, which is given by $\mathrm{P}=60-\mathrm{Q}$ ( P is the price and Q is industry output). Both firms will have the same cost function characterized by a constant marginal cost equal to 12 and zero fixed cost. Ms Moove is too busy to run her firm and is going to hire a manager to run the firm on her behalf. She has to decide what type of contract to offer to the manager. She is only considering two types of contract: a profit-sharing contract that will give the manager the fraction $\alpha$ of the firm's profits (and no fixed salary) or a revenue-sharing contract that will give the manager the fraction $\alpha$ of the firm's revenue (and no fixed salary). The parameter $\alpha$ is strictly between 0 and $\frac{1}{2}$ and is determined by social customs; in particular, Ms Moove cannot choose the value of $\alpha$ : she can only choose between a profit-sharing and a revenue-sharing contract. The contract with the manager is signed before competition takes place and is made public, so that - in particular - firm 1 will be aware of it. The owner of firm 1 is running his own firm and both he and Ms Moove are interested in maximizing their own income. The manager appointed by Ms Moove makes all the decisions concerning the firm and his objective is to maximize his own income.
(a) [18 points] What type of contract will Ms Moove offer her manager? [Hint: use the notion of SPE.]
(b) [5 points] What is Ms Moove's income if $\alpha=\frac{1}{16}$ ?
(c) [5 points] What is Ms Moove's income if $\alpha=\frac{1}{8}$ ?
4. [20 points]. In the following game, Nature moves first and chooses, with equal probability, one of two types ( $t_{1}$ or $t_{2}$ ) of player 1. Player 1 observes her own type and decides whether to choose L or R . If player 1 chooses R , the game ends; if player 1 chooses L , player 2 has to choose between T and B (without knowing the type of player 1). Payoffs are denoted by $(n, m)$ where $n$ is Player 1's payoff and $m$ is Player 2's payoff. Find all the pure-strategy weak sequential equilibria.


1. (a) It is the Vickrey or second-price auction.
(b) The weakly dominant strategy is to bid $V$. Fix a player $i$ and let $M$ be the $m^{\text {th }}$ bid on the seller's list modified by removing the bid of player $i$. Three cases are possible. Case $1: M<\mathrm{V}$. In this case, by bidding $V$ player $i$ obtains one unit and pays a price of $M$ (which is now the $(m+1)^{\text {th }}$ bid) and thus obtains a payoff of $V-M>0$. The same happens if he chooses any other bid $b_{i}>M$. If he chooses $b_{i}=$ $M$ then he either gets the same payoff as above or a payoff of zero (in case his index is higher than the index of the other player who submitted a bid of $M$ ). If he chooses $b_{i}<M$ then he does not get the object and his payoff is zero. Case 2: $M=V$. In this case, if he gets one unit he pays $V$ and thus his payoff is zero; if he doesn't get the object, his payoff is zero. Thus, any bid gives him a payoff of zero. Case 2: $M>V$. In this case, if he bids $V$ (or any other $b_{i}<M$ ) he does not get the good and his payoff is zero. If he chooses $b_{i}>M$ then he gets the object by paying $M$ and his payoff is $V-M<0$. If he chooses $b_{i}=M$, then he either does not get the object (in case his index is higher than the index of the other player who submitted a bid of $M$ ) and his payoff is zero, or he does get the object and pays $M$, obtaining a payoff of $V-M<0$.
Thus in all cases bidding $V$ is at least as good as submitting a different bid.
(c) Consider the case where everybody else bids less than $V$, say $V-\varepsilon$ (with $0<\varepsilon \leq V$ ), then by bidding $V$ player $i$ gets one unit at the price of $V-\varepsilon$ (and thus obtains a payoff of $\varepsilon>0$ ), but he can also get the unit, at the same price, by submitting a bid of $V-\frac{\varepsilon}{2}$ or a bid of $2 V$ (or any other bid higher than $V-\varepsilon$ ).
(d) The seller sells $m$ units at a price of $V$, thus her revenue is $m V$.
2. 

(a) The game is as follows

(b) The strategic form is as follows:

Player 2

|  | A |  | $B$ |  |
| :---: | :---: | :---: | :---: | :---: |
| $A$ | $\mathrm{b}-\mathrm{c}$, | $(1-r)(b-c)$ | rb-c, | $(1-r) b$ |
| $B$ | (1-r)b, | -(1-r) C | 0 , | 0 |

(c) For Player $1 B$ is a strictly dominant strategy if and only if $r<\frac{c}{b}$ and $A$ is a strictly dominant strategy if and only if $r>\frac{c}{b}$. (If $r=\frac{c}{b}$ then $A$ and $B$ are equivalent.)
(d) For Player $2 B$ is a strictly dominant strategy for every value of $r$.
(e) If $r<\frac{c}{b}$ the unique Nash equilibrium is $(B, B)$. If $r>\frac{c}{b}$ the unique Nash equilibrium is $(A, B)$. If $r=\frac{c}{b}$ then, for every $p \in[0,1],\left(\left(\begin{array}{cc}A & B \\ p & 1-p\end{array}\right), B\right)$ is a Nash equilibrium.
3. (a) If the manager is offered a profit-sharing contract then he will choose $q_{2}$ to maximize profits. Thus $q_{1}$ will be chosen to maximize $\Pi_{1}\left(q_{1}, q_{2}\right)=q_{1}\left(60-q_{1}-q_{2}\right)-12 q_{1}$ and $q_{2}$ will be chosen to maximize $\Pi_{2}\left(q_{1}, q_{2}\right)=q_{2}\left(60-q_{1}-q_{2}\right)-12 q_{2}$. Solving $\frac{\partial \Pi_{1}}{\partial q_{1}}=0$ and $\frac{\partial \Pi_{2}}{\partial q_{2}}=0$ gives $q_{1}=q_{2}=16$. Ms Moove's income will be $(1-\alpha) \Pi_{2}(16,16)=(1-\alpha) 256$.
If the manager is offered a revenue-sharing contract then he will choose $q_{2}$ to maximize revenue. Thus $q_{1}$ will be chosen to maximize $\Pi_{1}\left(q_{1}, q_{2}\right)=q_{1}\left(60-q_{1}-q_{2}\right)-12 q_{1}$ and $q_{2}$ will be chosen to maximize $R_{2}\left(q_{1}, q_{2}\right)=q_{2}\left(60-q_{1}-q_{2}\right)$. Solving $\frac{\partial \Pi_{1}}{\partial q_{1}}=0$ and $\frac{\partial R_{2}}{\partial q_{2}}=0$ gives $q_{1}=12$ and $q_{2}=24$. Ms Moove's income will be $\Pi_{2}(12,24)-\alpha R_{2}(12,24)=288-\alpha 576$. Now, $(1-\alpha) 256>288-\alpha 576$ if and only if $\alpha>\frac{1}{10}$. Thus she will choose a revenue-sharing contract if $\alpha<\frac{1}{10}$ and a profit-sharing contract if $\alpha>\frac{1}{10}$ $\begin{array}{llll}\text { (and be indifferent if } \alpha=0.1 \text { ). } & \text { (b) } 288-\frac{1}{16} 576=252 . & \text { (c) } \frac{7}{8} 256=224 .\end{array}$
4. Denote a pure strategy of Player 1 by $x y$, where $x$ is his choice if his type is $t_{1}$ and $y$ is his choice if his type is $t_{2}$. Furthermore, denote a belief for Player 2 by $\left(\begin{array}{cc}w & z \\ p & 1-p\end{array}\right)$ where $w$ is the top node (following choice $L$ by type $t_{1}$ ) of her information set and $z$ is the bottom node (following choice $L$ by type $t_{2}$ ). First note that if Player 2 attaches positive probability to node $w(p>0)$ then her unique best reply is $T$ and if Player 2 plays $T$ then the unique best reply of type $t_{2}$ of Player 1 is $L$, while type $t_{1}$ of Player 1 is indifferent between $L$ and $R$. Thus there are four pure-strategy weak sequential equilibria: (1) (LL,T) with $\left(\begin{array}{cc}w & z \\ 0.5 & 0.5\end{array}\right)$, (2) (RL,B) with $\left(\begin{array}{ll}w & z \\ 0 & 1\end{array}\right)$, (3) (RL,T) with $\left(\begin{array}{ll}w & z \\ 0 & 1\end{array}\right)$, (4) (RR,B) with $\left(\begin{array}{ll}w & z \\ 0 & 1\end{array}\right)$. On the other hand, Player 1's strategy LR cannot be part of a sequential equilibrium.

## FINAL EXAM

ANSWER ALL QUESTIONS (total 100 points)

1. [32 points] Let $e \in[0, \infty)$ denote the amount of education. There are two types of potential workers: those with productivity $\theta_{A}$ and those with productivity $\theta_{B}$, with $\theta_{B}>\theta_{A}>0$. Each potential worker knows what her productivity is, while the potential employer does not. For each type $i \in\{A, B\}$ the cost of acquiring $e$ units of education is $e \theta_{i}$. The fraction of type $B$ in the population is $\mu_{B} \in(0,1)$.
(a) [8 points] Is there a separating equilibrium where type $A$ chooses education level $e_{A}$, type $B$ chooses education level $e_{B}$, with $e_{A} \neq e_{B}$, and the employer pays each worker a wage equal to the worker's productivity? If there is, please describe it. If not, please explain why not.
(b) (b.1) $[8$ points] Suppose that the employer offers the following wage schedule (AP means "average productivity"): $\left\{\begin{array}{ll}\mathrm{AP} & \text { if } e=e^{*} \\ \theta_{A} & \text { if } e \neq e^{*}\end{array}\right.$. Find necessary and sufficient conditions for a pooling equilibrium, that is, an equilibrium where both types make the same education choice.
(b.2) $\left[4\right.$ points]Find all the pooling equilibria when $\mu_{B}=\frac{2}{5}, \theta_{A}=1$ and $\theta_{B}=6$.
(c) (c.1) $[8$ points $]$ Suppose that the employer offers the following wage schedule (where
$\left.0<e^{*}<\hat{e}\right):\left\{\begin{array}{cc}\theta_{A} & \text { if } e<e^{*} \\ \operatorname{AP} & \text { if } e^{*} \leq e<\hat{e} . \text {. Find necessary and sufficient conditions for a } \\ \theta_{B} & \text { if } e \geq \hat{e}\end{array}\right.$
pooling equilibrium.
(c.2) [4 points] Find all the pooling equilibria when $\mu_{B}=\frac{2}{5}, \theta_{A}=1$ and $\theta_{B}=6$.
2. [ 34 points] Players 1 and 2 are contemplating signing a contract. There are four possible states of the world and, if signed, the contract will yield the following payoffs:

| state | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
| :---: | :---: | :---: | :---: | :---: |
| 1's payoff | -3 | 10 | -2 | 10 |
| 2's payoff | 3 | 1 | 2 | -2 |

Let $p$ and $q$ be two sentences that describe facts about the world. Let $\neg$ denote negation and $\wedge$ conjunction (thus, for example, $p \wedge \neg q$ means that fact $p$ is true while fact $q$ is false). The states are described as follows: | state | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| facts | $p \wedge q$ | $\neg p \wedge q$ | $p \wedge \neg q$ | $\neg p \wedge \neg q$ | .

If the two players do not sign a contract, then each gets a payoff of zero. All the payoffs are von Neumann-Morgenstern payoffs and the players are expected utility maximizers.
The following is common knowledge between the two players:

1. Player 1 is uncertain about all the facts and has the following beliefs: $\left(\begin{array}{cccc}\alpha & \beta & \gamma & \delta \\ \frac{2}{6} & \frac{1}{6} & \frac{2}{6} & \frac{1}{6}\end{array}\right)$.
2. Player 2 knows whether or not $p$ is true (that is, if $p$ is true then player 2 knows that $p$ is true and $\mathrm{f} p$ is not true then player 2 knows that $p$ is not true). However, Player 2 is uncertain as to whether $q$ is true or not.
3. There exists a common prior for the beliefs of the players.

Player 1 moves first and decides whether or not to sign the contract. If she does not sign the contract, then the deal is off. If she signs the contract, then player 2 has to decide whether or not to sign. If he doesn't sign, the deal is off. If he signs the contract becomes binding.
(a) [6 points] What are the (conditional) beliefs of player 2 concerning the states?
(b) $[18$ points $]$ Represent this situation of incomplete information as a game of imperfect information.
(c) [10 points] Find all the pure-strategy weak sequential equilibria. Explain your answer.
3. [20 points] Consider a competitive insurance market with free entry where there are two types of potential customers, H and L . They all have the same initial wealth $W=\$ 15,000$ and the same potential loss $x=\$ 6,000$. The probability of loss for $H$ type is $p_{H}=25 \%$ and the probability of loss for $L$ type is $p_{L}=10 \%$. Let $q_{H}$ be the proportion of $H$ types in the population. They all have the same, strictly concave, utility-of-wealth function $U(m)$. Define a separating equilibrium as a pair of contracts $\left(\mathrm{C}_{\mathrm{H}}, \mathrm{C}_{\mathrm{L}}\right)$ such that (1) type H individuals purchase contract $\mathrm{C}_{\mathrm{H}}$, which they find at least as good as contract $\mathrm{C}_{\mathrm{L}}$ and at least as good as no insurance, (2) type L individuals purchase contract $\mathrm{C}_{\mathrm{L}}$, which they find at least as good as contract $\mathrm{C}_{\mathrm{H}}$ and at least as good as no insurance, (3) each contract yields zero expected profits, and (4) no insurance company could make positive profits by introducing a new contract.
(a) [ 9 points] Draw a diagram showing the equilibrium (using indifference curves, etc.), if it exists.
(b) [8 points] Write equations whose solution gives the pair of contracts that constitutes an equilibrium, if an equilibrium exists.
(c) $[3$ points $]$ Explain what additional condition needs to be satisfied for the pair of contracts of part (b) to be an equilibrium (no need to do any math, just explain in words)
4. [14 points] Consider the following two-player game, where the payoffs are von Neumann-Morgenstern payoffs:

## Player 2



However, the game is not played simultaneously, but as follows. At date 0, Player 1 commits to a (possibly degenerate) mixed strategy. At date 1 Player 2 observes Player 1's commitment and chooses a (possibly degenerate) mixed strategy. At date 3 the previously chosen mixed strategies are implemented and payoffs are realized. Find the subgame-perfect equilibrium payoffs of this game, assuming that the probabilities assigned to the pure strategies must be integer multiples of 0.05 , that is, must belong to the set $\{0,0.05,0.10,0.15, \ldots, 0.95,1\}$. There is no discounting.

## Final Exam ANSWERS

1. (a) Suppose there is a signaling equilibrium where type $A$ chooses education level $e_{A}$ and is paid $\theta_{A}$ and type $B$ chooses education level $e_{B}$ and is paid $\theta_{B}$, with $e_{A} \neq e_{B}$. Then the following incentive compatibility constraints must be satisfied (the first is type $A$ 's and the second type $B$ 's):

$$
\begin{align*}
& \theta_{A}-e_{A} \theta_{A} \geq \theta_{B}-e_{B} \theta_{A}  \tag{1}\\
& \theta_{B}-e_{B} \theta_{B} \geq \theta_{A}-e_{A} \theta_{B} \tag{2}
\end{align*}
$$

Adding (1) and (2) and simplifying we get that $e_{A}\left(\theta_{B}-\theta_{A}\right) \geq e_{B}\left(\theta_{B}-\theta_{A}\right)$ from which it follows (since $\theta_{B}-\theta_{A}>0$ and $e_{A} \neq e_{B}$ ) that $e_{A}>e_{B}$. But then (1) is violated because
$\theta_{B}-e_{B} \theta_{A} \underset{\text { since } e_{A}>e_{B}}{>} \theta_{B}-e_{A} \theta_{A} \underset{\text { since } \theta_{A}<\theta_{B}}{>} \theta_{A}-e_{A} \theta_{A}$.
(b) (b.1) Let us look for a pooling equilibrium where both types choose $e=e^{*}$. Let $\bar{\theta}=\mu_{B} \theta_{B}+\left(1-\mu_{B}\right) \theta_{A}$. The incentive compatibility constraints are:

$$
\begin{array}{cl}
\bar{\theta}-e^{*} \theta_{A} \geq \theta_{A}-e \theta_{A}, & \forall e \in[0, \infty) \\
\bar{\theta}-e^{*} \theta_{B} \geq \theta_{A}-e \theta_{B}, & \forall e \in[0, \infty) \tag{2}
\end{array}
$$

Since the RHS of (1) and (2) is decreasing in $e$ we can rewrite (1) as $\bar{\theta}-e^{*} \theta_{A} \geq \theta_{A}$ and (2) as $\bar{\theta}-e^{*} \theta_{B} \geq \theta_{A}$. If the latter is satisfied then so is the former, since $\theta_{A}<\theta_{B}$. Thus the necessary and sufficient condition for a pooling equilibrium where both types choose $e=e^{*}$ is $\bar{\theta}-e^{*} \theta_{B} \geq \theta_{A}$, that is, $e^{*} \leq \frac{\bar{\theta}-\theta_{A}}{\theta_{B}}$. Another possibility is a pooling equilibrium where both types choose $e=0$. In this case the incentive compatibility constraints are as follows (since choosing $e$ such that $0<\mathrm{e}<\mathrm{e}^{*}$ is strictly dominated by choosing $e=0$ and choosing $e$ such that $\mathrm{e}>\mathrm{e}^{*}$ is strictly dominated by choosing $e=e^{*}$ ):

$$
\begin{align*}
& \theta_{A} \geq \bar{\theta}-e^{*} \theta_{A} \\
& \theta_{A} \geq \bar{\theta}-e^{*} \theta_{B}
\end{align*}
$$

Since $\theta_{A}<\theta_{B}$, if $\left(1^{\prime}\right)$ is satisfied then so is $\left(2^{\prime}\right)$. Thus a necessary and sufficient condition for a pooling equilibrium where both types choose $e=0$ is $e^{*} \geq \frac{\bar{\theta}}{\theta_{A}}-1$.
(b.2) When $\mu_{B}=\frac{2}{5}, \theta_{A}=1$ and $\theta_{B}=6$ we get that $\bar{\theta}=\frac{2}{5} 6+\frac{3}{5} 1=3$ and thus, using the calculations of part (b.1) we conclude that any $e^{*} \leq \frac{3-1}{6}=\frac{1}{3}$ gives rise to a pooling equilibrium where both types choose $e=e^{*}$ and any $e^{*} \geq 2$ gives rise to a pooling equilibrium where both types choose $e=0$.
(c) (c.1) Let us look for a pooling equilibrium where both types choose $e=e^{*}$. The incentive compatibility constraints are:

$$
\begin{gather*}
\bar{\theta}-e^{*} \theta_{A} \geq \theta_{A}-e \theta_{A}, \quad \forall e \in\left[0, e^{*}\right)  \tag{1a}\\
\bar{\theta}-e^{*} \theta_{A} \geq \theta_{B}-e \theta_{A}, \quad \forall e \in[\hat{e}, \infty)  \tag{1b}\\
\bar{\theta}-e^{*} \theta_{B} \geq \theta_{A}-e \theta_{B}, \quad \forall e \in\left[0, e^{*}\right)  \tag{2a}\\
\bar{\theta}-e^{*} \theta_{B} \geq \theta_{B}-e \theta_{B}, \quad \forall e \in[\hat{e}, \infty) \tag{2b}
\end{gather*}
$$

Since the RHS of each inequality is decreasing in $e$, we can rewrite them as

$$
\begin{align*}
& \bar{\theta}-e^{*} \theta_{A} \geq \theta_{A}  \tag{1a}\\
& \bar{\theta}-e^{*} \theta_{A} \geq \theta_{B}-\hat{e} \theta_{A}  \tag{1b}\\
& \bar{\theta}-e^{*} \theta_{B} \geq \theta_{A}  \tag{2a}\\
& \bar{\theta}-e^{*} \theta_{B} \geq \theta_{B}-\hat{e} \theta_{B} \tag{2b}
\end{align*}
$$

First of all, note that - since $\theta_{A}<\theta_{B}-(2 \mathrm{a})$ implies (1a). Thus we only need to consider the remaining inequalities, which can be re-written as follows:

$$
\begin{align*}
& \left(\hat{e}-e^{*}\right) \theta_{A} \geq \theta_{B}-\bar{\theta}  \tag{1b}\\
& \bar{\theta} \geq \theta_{A}+e^{*} \theta_{B}  \tag{2a}\\
& \left(\hat{e}-e^{*}\right) \theta_{B} \geq \theta_{B}-\bar{\theta} \tag{2b}
\end{align*}
$$

Since $\hat{e}>e^{*}$ and $\theta_{B}>\theta_{A},\left(\hat{e}-e^{*}\right) \theta_{B}>\left(\hat{e}-e^{*}\right) \theta_{A}$ and thus (1.b) implies (2.b). Thus we only need to consider the two inequalities

$$
\begin{align*}
& \left(\hat{e}-e^{*}\right) \theta_{A} \geq \theta_{B}-\bar{\theta}  \tag{1b}\\
& \bar{\theta} \geq \theta_{A}+e^{*} \theta_{B} \tag{2a}
\end{align*}
$$

Since $\bar{\theta}>\theta_{A}$ inequality (2a) can be satisfied if $e^{*}$ is sufficiently close to 0 . Furthermore, if ( $\hat{e}-e^{*}$ ) is sufficiently large then also (1b) is satisfied. Thus a pooling equilibrium where both types choose $e=e^{*}$ can exist. For example, if $\mu_{B}=\frac{2}{5}, \theta_{A}=1$ and $\theta_{B}=6$ so that $\bar{\theta}=\frac{2}{5} 6+\frac{3}{5} 1=3$, then any pair $\left(e^{*}, \hat{e}\right)$ such that $e^{*} \leq \frac{1}{3}$ and $\hat{e} \geq e^{*}+3$.
Now let us look for a pooling equilibrium where both types choose $e=0$. Then the incentive compatibility constraints are:

$$
\begin{align*}
& \theta_{A} \geq \bar{\theta}-e^{*} \theta_{A}  \tag{1a}\\
& \theta_{A} \geq \theta_{B}-\hat{e} \theta_{A}  \tag{1b}\\
& \theta_{A} \geq \bar{\theta}-e^{*} \theta_{B}  \tag{2a}\\
& \theta_{A} \geq \theta_{B}-\hat{e} \theta_{B} \tag{2b}
\end{align*}
$$

Since $\theta_{A}<\theta_{B}$, (1a) implies (2a) and (1b) implies (2b). Thus a necessary and sufficient condition is $\left.\theta_{A} \geq \operatorname{Max}\left\{\bar{\theta}-e^{*} \theta_{A}, \theta_{B}-\hat{e} \theta_{A}\right)\right\}$, that is, $e^{*} \geq \frac{\bar{\theta}}{\theta_{A}}-1$ and $\hat{e} \geq \frac{\theta_{B}}{\theta_{A}}-1$.
One could also look for necessary and sufficient conditions for a pooling equilibrium where both types choose $e=\hat{e}$. The logic is the same.
(c.2) When $\mu_{B}=\frac{2}{5}, \theta_{A}=1$ and $\theta_{B}=6$ we get that $\bar{\theta}=\frac{2}{5} 6+\frac{3}{5} 1=3$ and thus a sufficient condition for a pooling equilibrium is $e^{*} \geq 2$ and $\hat{e} \geq 5$.
2. (a) The common prior must coincide with player 1 's beliefs. Thus, since the information partition of player 2 is $\{\{\alpha, \gamma\},\{\beta, \delta\}\}$, his conditional beliefs are obtained by using Bayes' rule and are given by $\left(\begin{array}{cc}\alpha & \gamma \\ \frac{1}{2} & \frac{1}{2}\end{array}\right)$ and $\left(\begin{array}{cc}\beta & \delta \\ \frac{1}{2} & \frac{1}{2}\end{array}\right)$.
(b) The extensive form is as follows (note that the states are ordered as $\alpha, \gamma, \beta$ and $\delta$ ).

(c) At the information on the left, $s$ strictly dominates $n$ for player 2 . Thus sequential rationality dictates the choice of $s$ there. Suppose that there is an equilibrium where player 1 chooses $s$ with positive probability. Then, by Bayes' rule, player 2's beliefs at the information set on the right must be probability $\frac{1}{2}$ on each node with the consequence that $s$ gives player 2 an expected payoff of $-\frac{1}{2}$, so that sequential rationality requires player 2 to choose $n$ there. Hence Player 1's payoff from playing $s$ is $\frac{2}{6}(-3)+\frac{2}{6}(-2)+\frac{1}{6}(0)+\frac{1}{6}(0)=-\frac{10}{6}$, while $n$ gives a payoff of 0 . Thus it cannot be optimal for 1 to play $s$ with positive probability. Hence in equilibrium player 1 plays $n$ with probability 1 . The following are all the pure-strategy weak sequential equilibria: player 1 plays $n$ with probability 1 and has beliefs $\left(\begin{array}{cccc}x_{\alpha} & x_{\beta} & x_{\gamma} & x_{\delta} \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{2}{6}\end{array}\right)$ (where $x_{i}$ is the node following state $i$ ); player 2 plays $s$ with probability 1 at the left information set and $n$ with probability 1 at the right information set, with beliefs $(p, 1-p)$ at the left information set (arbitrary $p$ ) and for the information set on the right probability $q$ on the left node and ( $1-q$ ) on the right node with $q \leq \frac{2}{3}$. The constraint $q \leq \frac{2}{3}$ is necessary for $n$ to be sequentially rational at
the right information set of player 2. [If it were the case that $q>\frac{2}{3}$ then $s$ would be uniquely sequentially rational at that information set; but then player 1's payoff from playing $s$ would be $\frac{2}{6}(-3)+\frac{2}{6}(-2)+\frac{1}{6}(10)+\frac{1}{6}(10)=\frac{10}{6}>0$ (the payoff from playing $n$ is 0 ) so that player 1 would also want to play $s$, forcing (by Bayes' rule) $q=\frac{1}{2}$, yielding a contradiction.]
3. (a) The two contracts are represented as points $H$ and $L$ in the following figure (points $A$ and $P$ can be ignored):

(b) H is the full insurance contract on the zero-profit (or fair odds) line for type H . Thus $\mathbf{C}_{\mathbf{H}}$ is the full-insurance contract with premium $\mathbf{h}=\mathbf{\$ 1 , 5 0 0}$ ( $=\frac{1}{4} 6000$, the expected loss for the H type). The two equations that identify contract $\mathrm{C}_{\mathrm{L}}$ are: $U(13,500)=\frac{1}{4} U\left(15,000-h_{L}-d_{L}\right)+\frac{3}{4} U\left(15,000-h_{L}\right)$ and $h_{L}=\frac{1}{10}\left(6,000-d_{L}\right)$.
(c) it must be the case that the average fair odds line be below (or at most tangent to) the indifference curve of the $L$ type through contract $L$. This amounts to saying that the fraction of H type in the population is sufficiently high
4. Let $p$ be the probability of $U$. Then, if $p<0.5$ Player 2's best response is to choose the pure strategy $R$, if $p>0.5$ Player 2's best response is to choose the pure strategy $L$ and if $p=0.5$ then any mixed-strategy of Player 2 is a best response. Then Player 1's payoff is as follows:
$\pi_{1}(p)=\left\{\begin{array}{ll}3 p+2(1-p)=2+p & \text { if } p<0.5 \\ p & \text { if } p>0.5\end{array}\right.$, while if $p=0.5$ then it depends on the mixed-strategy
chosen by Player 2: letting $q$ be the probability of $L$ it is $\pi_{1}(0.5, q)=0.5[q+3(1-q)]+0.5[2(1-q)]=2.5-2 q$. Thus if Player 1 anticipates a response of $q>0$ to his choice of $p=0.5$, then he will choose $p=0.45$, while if he anticipates a response of $q=0$ to his choice of $p=0.5$, then he will choose $p=0.5$. The subgame-perfect equilibrium payoffs are: (1) $(2.5$, $0.5)$, associated with the choices of $p=0.5$ and the response $q=0$, and (2) $(2.45,0.55)$, associated with the choices of $p=0.45$ and the response $q=0$. [The full strategy of Player 2 is as explained above.]

