COURNOT DUOPOLY: an example

Let the inverse demand function and the cost function be given by

$$P = 50 - 2Q \quad \text{and} \quad C = 10 + 2q$$

respectively, where Q is total industry output and q is the firm's output.

First consider first the case of **uniform-pricing monopoly**, as a benchmark. Then in this case Q = q and the profit function is

$$\pi(Q) = (50 - 2Q)Q - 10 - 2Q = 48Q - 2Q^2 - 10.$$

Solving $\frac{d\pi}{dQ} = 0$ we get Q = 12, P = 26, $\pi = 278$, CS = $\frac{12(50-26)}{2}$ = 144, TS = 278 + 144 = 422.

MONOPOLY

Q	P	π	CS	TS
12	26	278	144	422

Now let us consider the case of two firms, or **duopoly**. Let q_1 be the output of firm 1 and q_2 the output of firm 2. Then $Q = q_1 + q_2$ and the profit functions are:

$$\pi_1(q_1,q_2) = q_1 [50 - 2 (q_1 + q_2)] - 10 - 2q_1$$

$$\pi_2(q_1,q_2) = q_2 [50 - 2 (q_1 + q_2)] - 10 - 2q_2$$

A Nash equilibrium is a pair of output levels (q_1^*, q_2^*) such that:

$$\pi_1(q_1^*, q_2^*) \geq \pi_1(q_1, q_2^*) \quad \text{for all } q_1 \geq 0$$

and

$$\pi_2(q_1^*, q_2^*) \geq \pi_1(q_1^*, q_2) \quad \text{for all } q_2 \geq 0.$$

This means that, fixing q_2 at the value q_2^* and considering π_1 as a function of q_1 alone, this function is maximized at q_1 = q_1^*. But a necessary condition for this to be true is that $\frac{\partial \pi_1}{\partial q_1}(q_1^*, q_2^*) = 0$. Similarly, fixing q_1 at the value q_1^* and considering π_2 as a function of q_2 alone, this function is maximized at q_2 = q_2^*. But a necessary condition for this to be true is that $\frac{\partial \pi_2}{\partial q_2}(q_1^*, q_2^*) = 0$. Thus the Nash equilibrium is found by solving the following system of two equations in the two unknowns q_1 and q_2:

$$\begin{cases} \frac{\partial \pi_1}{\partial q_1}(q_1^*, q_2^*) = 50 - 4q_1 - 2q_2 - 2 = 0 \\ \frac{\partial \pi_2}{\partial q_2}(q_1^*, q_2^*) = 50 - 2q_1 - 4q_2 - 2 = 0 \end{cases}$$

The solution is $q_1^* = q_2^* = 8$, Q = 16, P = 18, π_1 = π_2 = 118, CS = $\frac{16(50-18)}{2}$ = 256, TS = 118 + 118 + 256 = 492.

Let us compare the two.

MONOPOLY

Q	P	π	CS	TS
12	26	278	144	422

DUOPOLY

q_1	q_2	Q	P	π_1	π_2	tot π	CS	TS
8	8	16	18	118	118	236	256	492

Thus competition leads to an increase not only in consumer surplus but in total surplus: the gain in consumer surplus (256 – 144 = 112) exceeds the loss in total profits (278 – 236 = 42).

In the above example we assumed that the two firms had the same cost function (C = 10 + 2q). However, there is no reason why this should be true. The same reasoning applies to the case where the **firms have different costs**. Example: demand function as before (P = 50 – 2Q) but now

cost function of firm 1: $C_1 = 10 + 2q_1$

cost function of firm 2: $C_2 = 12 + 8q_2$.

Then the profit functions are:

$$\pi_1(q_1,q_2) = q_1\,[50 - 2\,(q_1 + q_2)] - 10 - 2q_1$$

$$\pi_2(q_1,q_2) = q_2\,[50 - 2\,(q_1 + q_2)] - 12 - 8q_2$$

The Nash equilibrium is found by solving:

$$\begin{cases} \dfrac{\partial \pi_1}{\partial q_1}(q_1^*, q_2^*) = 50 - 4q_1 - 2q_2 - 2 = 0 \\ \dfrac{\partial \pi_2}{\partial q_2}(q_1^*, q_2^*) = 50 - 2q_1 - 4q_2 - 8 = 0 \end{cases}$$

The solution is $q_1^* = 9,\ q_2^* = 6$, Q = 15, P = 20, π_1 = 152, π_2 = 60. Since firms have different costs, they choose different output levels: **the low-cost firm (firm 1) produces more** and makes higher profits **than the high-cost firm (firm 2).**

COURNOT OLIGOPOLY: too many firms

$a := 50 \quad b := 2 \quad c := 2 \quad F := 10$

Inverse demand $\quad P(Q) := a - b\cdot Q \quad \boxed{P(Q) \rightarrow 50 - 2\cdot Q}$ demand

Cost function: $\quad C(q) := F + c\cdot q \quad \boxed{C(q) \rightarrow 10 + 2\cdot q}$ cost

Profit function of firm 1: $\Pi_1(q_1,...,q_n) = q_1\,[50 - 2(q_1 + ... + q_n)] - 2\,q_1 - 10$

Derivative: $50 - 2(2q_1 + q_2 + ... + q_n) - 2$ Symmetric solution requires $q_1 = ... = q_n$

so we have $48 - 2(n+1)\,q = 0$ Thus

$$q(n) := \frac{48}{(n+1)\cdot 2} \qquad \boxed{q(n)\ \text{simplify} \rightarrow \frac{24}{(n+1)}}$$ firm output

$$Q(n) := n\cdot q(n) \qquad \boxed{Q(n)\ \text{simplify} \rightarrow 24\cdot\frac{n}{(n+1)}}$$ industry output

$$p(n) := 50 - 2\cdot n\cdot q(n) \qquad \boxed{p(n)\ \text{simplify} \rightarrow 2\cdot\frac{(n+25)}{(n+1)}}$$ price

$$(p(n) - 2)\ \text{simplify} \rightarrow \frac{48}{(n+1)}$$

$$\boxed{Pr(n) := \frac{48\cdot 24}{(n+1)^2} - 10}$$ PROFITS of each firm

$$Pr_{tot}(n) := n\cdot Pr(n)$$

$$CS(n) := \frac{(P(0) - p(n))\cdot Q(n)}{2} \qquad \boxed{CS(n)\ \text{simplify} \rightarrow 576\cdot\frac{n^2}{(n+1)^2}}$$ consumer surplus

$$SW(n) := CS(n) + Pr_{tot}(n) \qquad SW(n)\ \text{simplify} \rightarrow -2\cdot n\cdot\frac{\left(-278\cdot n - 571 + 5\cdot n^2\right)}{(n+1)^2}$$ social welfare

$n := 1, 2 .. 10$

$n =$	$Q(n) =$	$p(n) =$	$Pr(n) =$	$Pr_{tot}(n) =$	$CS(n) =$	$SW(n) =$
1	12	26	278	278	144	422
2	16	18	118	236	256	492
3	18	14	62	186	324	510
4	19.2	11.6	36.08	144.32	368.64	512.96
5	20	10	22	110	400	510
6	20.5714286	8.8571429	13.5102041	81.0612245	423.1836735	504.244898
7	21	8	8	56	441	497
8	21.3333333	7.3333333	4.2222222	33.7777778	455.1111111	488.8888889
9	21.6	6.8	1.52	13.68	466.56	480.24
10	21.8181818	6.3636364	-0.4793388	-4.7933884	476.0330579	471.2396694

Thus (free entry) equilibrium number of firms in the industry is 9.

The socially optimum number of firms is 4.