UNCERTAINTY in GAMES

A. Subjective or strategic uncertainty.

		Player 2	
		C	D
Player 1	A	z_1	z_2
	B	z_3	z_4

Suppose that **Player 1's ranking** of the outcomes is:

$z_1 \succ z_4 \succ z_3 \succ z_2$

best worst

If Player 1 believes that Player 2 will play C will play A

If Player 1 believes that Player 2 will play D " B

If Player 1 believes that Player 2 is equally likely to play C or D

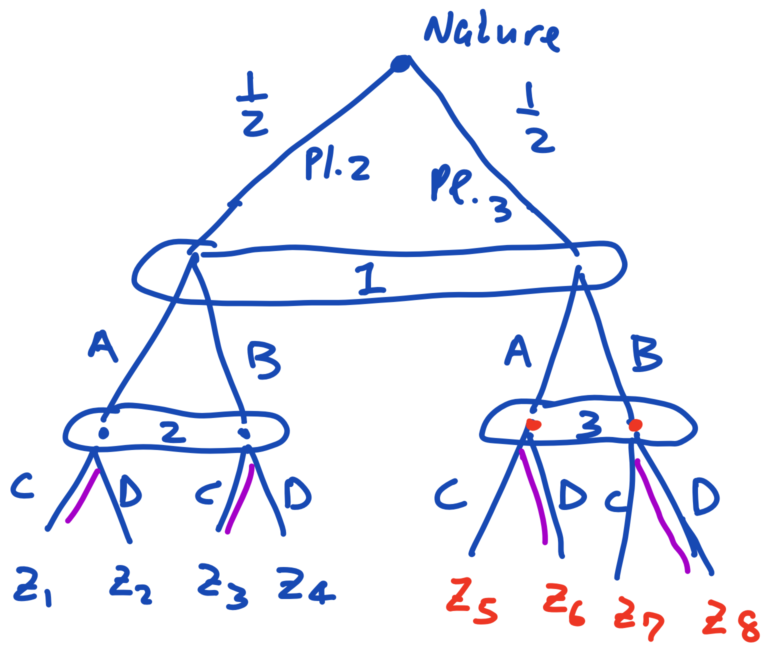
$$A \to \begin{pmatrix} z_1 & z_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$B \to \begin{pmatrix} z_3 & z_4 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

B. Objective uncertainty.

Player 1 has to choose between *A* and *B* knowing that she is playing a simultaneous game against another player, but not knowing whether it is Player 2 or Player 3. She is in front of a computer terminal. The computer will “toss a coin” and choose the opponent (and inform the opponent, but not Player 1).

$A \to \begin{pmatrix} z_1 & z_6 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad B \to \begin{pmatrix} z_3 & z_8 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$



Player 3's preferences:

$\underbrace{z_6 \succ_3 z_5}_{\text{at left node D better than C}}$ and $\underbrace{z_8 \succ_3 z_7}_{\text{at right node D better than C}}$

predict that Player 3 will play D

Player 2's preferences: $\underbrace{z_1 \succ_2 z_2}_{\text{at left node C better than D}}$ and $\underbrace{z_3 \succ_2 z_4}_{\text{at right node C better than D}}$

predict that Player 2 would choose C

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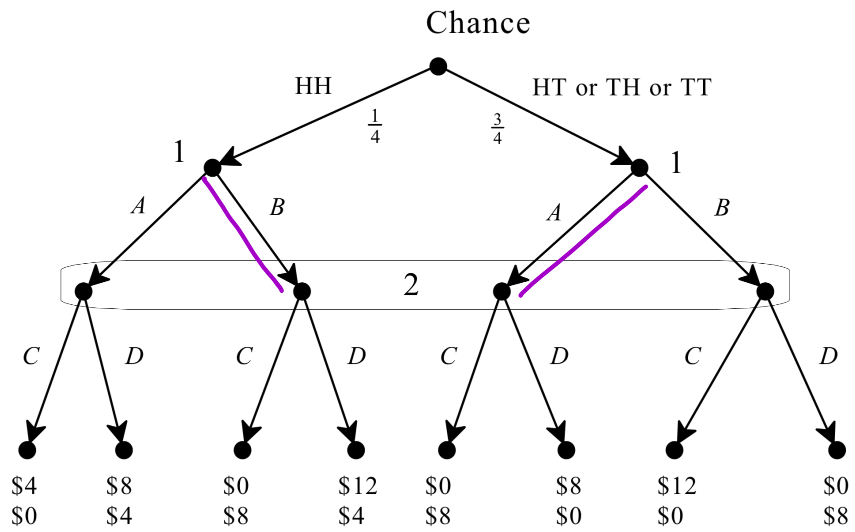
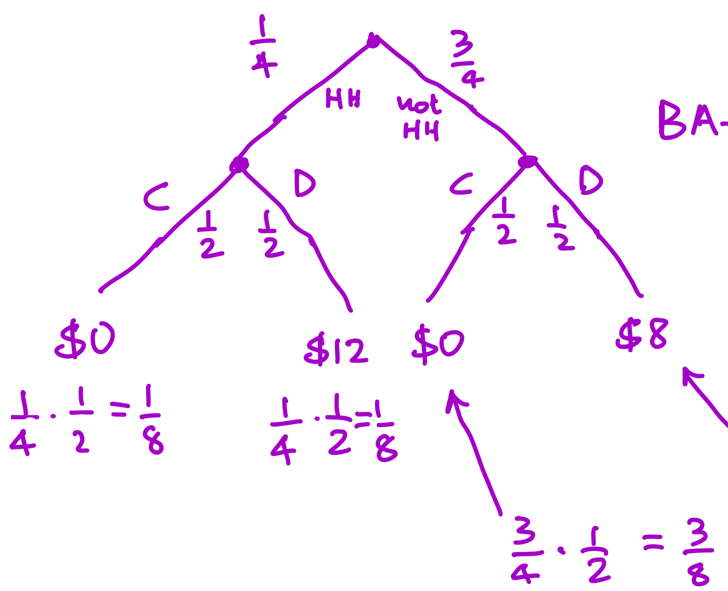


Player 1 has 4 strategies :

AA, AB, BA, BB

Assume that Player 1 is selfish and greedy

Suppose that " thinks that Player 2 is equally likely to play C and D



$$BA \to \underbrace{\begin{pmatrix} \$0 & \$8 & \$12 \\ \frac{1}{8}+\frac{3}{8}=\frac{4}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}}$$

if Player 1 is risk neutral then this lottery is equivalent to

$$\frac{4}{8}0 + \frac{3}{8}8 + \frac{1}{8}12 =$$

$$= \frac{36}{8}$$

$$\begin{pmatrix} \$1M \\ 1 \end{pmatrix} \text{ versus } \begin{pmatrix} \$0 & \$2M \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$Z = \{z_1, z_2, \ldots, z_n\}$$ set of basic outcomes

A lottery $L = \begin{pmatrix} z_1 & z_2 & \cdots & z_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$ $0 \le p_i \le 1$ for all $i = 1, \ldots, n$

$p_1 + p_2 + \cdots + p_n = 1$

denote $\begin{pmatrix} z_1 & z_2 & \cdots & z_n \\ 0 & 1 & & 0 \end{pmatrix}$ by $\begin{pmatrix} z_2 \\ 1 \end{pmatrix}$

" $\begin{pmatrix} z_1 & z_2 & \cdots & z_n \\ \frac{1}{3} & 0 & & \frac{2}{3} \end{pmatrix}$ by $\begin{pmatrix} z_1 & z_n \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

Theorem 1 Let $Z = \{z_1, z_2, ..., z_m\}$ be a set of basic outcomes and $\mathcal{L}$ the set of lotteries over *Z*. If $\succsim$ is a von Neumann-Morgenstern ranking of the elements of $\mathcal{L}$ then there exists a function U, called a *von Neumann-Morgenstern utility function*, that assigns a number (called utility) to every basic outcome and is such that, for any two lotteries $L = \begin{pmatrix} z_1 & z_2 & \cdots & z_m \\ p_1 & p_2 & \cdots & p_m \end{pmatrix}$ and $M = \begin{pmatrix} z_1 & z_2 & \cdots & z_m \\ q_1 & q_2 & \cdots & q_m \end{pmatrix}$,

$$L \succ M \quad \text{if and only if} \quad \underbrace{p_1U(z_1) + p_2U(z_2) + ... + p_mU(z_m)}_{\text{expected utility of lottery } L} > \underbrace{q_1U(z_1) + q_2U(z_2) + ... + q_mU(z_m)}_{\text{expected utility of lottery } M}$$

and

$$L \sim M \quad \text{if and only if} \quad \underbrace{p_1U(z_1) + p_2U(z_2) + ... + p_mU(z_m)}_{\text{expected utility of lottery } L} = \underbrace{q_1U(z_1) + q_2U(z_2) + ... + q_mU(z_m)}_{\text{expected utility of lottery } M}$$

$m = 4$

2.25 3.33

EXAMPLE 1. $Z = \{z_1, z_2, z_3, z_4\}$ $L = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{8} & \frac{5}{8} & 0 & \frac{2}{8} \end{pmatrix}$ $M = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{2}{6} \end{pmatrix}$

Suppose we know that $U = \begin{cases} z_1 & z_2 & z_3 & z_4 \\ 6 & 2 & 8 & 1 \end{cases}$

$$E[U(L)] = \frac{1}{8} \cdot \underset{6}{U(z_1)} + \frac{5}{8} \underset{2}{U(z_2)} + 0\, \underset{8}{U(z_3)} + \frac{2}{8} \underset{1}{U(z_4)}$$

$$= \frac{1}{8} 6 + \frac{5}{8} 2 + \frac{2}{8} \cdot 1 = 2.25$$

$$E[U(M)] = \frac{1}{6} \underset{6}{U(z_1)} + \frac{2}{6} \underset{2}{U(z_2)} + \frac{1}{6} \underset{8}{U(z_3)} + \frac{2}{6} \underset{1}{U(z_4)}$$

$$= \frac{1}{6} 6 + \frac{2}{6} \cdot 2 + \frac{1}{6} 8 + \frac{2}{6} \cdot 1 = 3.33$$

Since $3.33 > 2.25$ $M \succ L$

EXAMPLE 2.

$$A = \begin{pmatrix} \overbrace{\text{paid 3-week vacation}}^{z_1} & \overbrace{\text{no vacation}}^{z_2} \\ 50\% & 50\% \end{pmatrix} \qquad B = \begin{pmatrix} \overbrace{\text{paid 1-week vacation}}^{z_3} \\ 100\% \end{pmatrix}$$

Suppose Ann says $\boxed{B \succ A}$ How would she rank

$$C = \begin{pmatrix} \text{paid 3-week vacation} & \text{no vacation} \\ 5\% & 95\% \end{pmatrix} \text{ and } D = \begin{pmatrix} \text{paid 1-week vacation} & \text{no vacation} \\ 10\% & 90\% \end{pmatrix} ?$$

best z_1 a

z_3 b

worst z_2 c

$a > b > c$

Suppose that this person chooses B:

$B \succ A$

Suppose he chooses C : $C \succ D$

From $B \succ A$ we infer that

$E[U(B)] > E[U(A)]$

$1 \cdot b > \frac{1}{2} a + \frac{1}{2} c$ $\qquad b > \frac{a+c}{2}$

or $\boxed{2b > a+c}$

From $C \succ D$ we infer that $E[U(C)] > E[U(D)]$

$$\frac{5}{100} \cdot a + \frac{95}{100} c > \frac{10}{100} b + \frac{90}{100} c$$

$$5a + 95c > 10b + 90c$$

$$5a + 5c > 10b$$

$$\boxed{a + c > 2b}$$

contradiction

Theorem 2. Let $\succsim$ be a von Neumann-Morgenstern ranking of the set of basic lotteries $\mathcal{L}$. Then the following are true.

(A) If $U : Z \to \mathbb{R}$ is a von Neumann-Morgenstern utility function that represents $\succsim$, then, for any two real numbers a and b with $a > 0$, the function $V : Z \to \mathbb{R}$ defined by $V(z_i) = aU(z_i) + b$ $(i = 1,2,\ldots,m)$ is also a von Neumann-Morgenstern utility function that represents $\succsim$.

(B) If $U : Z \to \mathbb{R}$ and $V : Z \to \mathbb{R}$ are two von Neumann-Morgenstern utility functions that represent $\succsim$, then there exist two real numbers a and b with $a > 0$ such that $V(z_i) = aU(z_i) + b$ $(i = 1,2,\ldots,m)$.

$$U = \begin{cases} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ 10 & 6 & 16 & 8 & 6 & 14 \end{cases}$$

$a = 2$

$$V = \begin{cases} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ 20 & 12 & 32 & 16 & 12 & 28 \end{cases}$$

$\downarrow$ $b = -10$

$$W = \begin{cases} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ 10 & 2 & 22 & 6 & 2 & 18 \end{cases}$$

U just as good as V just as good as W as a representation of an individual's preferences over lotteries over Z

Theorem 2. Let $\succsim$ be a von Neumann-Morgenstern ranking of the set of basic lotteries $\mathcal{L}$. Then the following are true.

(A) If $U : Z \to \mathbb{R}$ is a von Neumann-Morgenstern utility function that represents $\succsim$, then, for any two real numbers a and b with $a > 0$, the function $V : Z \to \mathbb{R}$ defined by $V(z_i) = aU(z_i) + b$ ($i = 1,2,\ldots,m$) is also a von Neumann-Morgenstern utility function that represents $\succsim$.

(B) If $U : Z \to \mathbb{R}$ and $V : Z \to \mathbb{R}$ are two von Neumann-Morgenstern utility functions that represent $\succsim$, then there exist two real numbers a and b with $a > 0$ such that $V(z_i) = aU(z_i) + b$ ($i = 1,2,\ldots,m$).

$$U = \begin{cases} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ 10 & 6 & 16 & 8 & 6 & 14 \end{cases}$$

$b = -6$

$$\tilde{V} = \begin{array}{cccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ 4 & 0 & 10 & 2 & 0 & 8 \end{array}$$

$a = \frac{1}{10}$

$$\tilde{W} = \begin{array}{cccccc} \frac{4}{10} & 0 & 1 & \frac{2}{10} & 0 & \frac{8}{10} \end{array}$$

← normalized utility function (value 1 for best " 0 for worst)

(10,6,2,0)->A,
(10,8,4,0)->B

		Player 2	
		C	D
Player 1	A	z_1	z_2
	B	z_3	z_4

Suppose that **Player 1's ranking** of the outcomes is:

$z_1 \succ z_4 \succ z_3 \succ z_2$

best ... worst

suppose answer is

Q. 1 : how do rank the outcomes $z_1, \dots, z_4$?

best	z_1	1
	z_4	0.6
	z_3	
worst	z_2	0

[Continuity axiom]

Q. 2 : if you have to choose between $\begin{pmatrix} z_4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} z_1 & z_2 \\ p & 1-p \end{pmatrix}$

what value of p would make you indifferent ?

Suppose answer is : $p = 0.6$

$$\begin{pmatrix} z_4 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_1 & z_2 \\ 0.6 & 0.4 \end{pmatrix}$$

$$EU = 1 \cdot U(z_4) \quad = \quad EU \quad 0.6\,\underset{1}{U(z_1)} + 0.4\,\underset{0}{U(z_2)}$$

$$= 0.6 \cdot 1 + 0.4 \cdot 0$$

$$= 0.6$$