COOPERATIVE GAMES: the CORE

So far we have looked at non-cooperative games, characterized by the fact that the individuals involved cannot sign binding agreements and therefore any suggested outcome has to be self-enforcing (i.e. a Nash equilibrium) for the players to be willing to go along with it. We now turn to **cooperative** games, where **binding agreements** are possible. The central question then becomes: *what agreement would the individuals involved be willing to subscribe to?*

The description of a cooperative game is in terms of a **characteristic function** which specifies for every group of players (i.e. every group of individuals who might enter into a binding agreement) the total payoff (e.g. money) that the members of S can obtain by signing an agreement among themselves; this payoff is available for distribution among the members of the group.

DEFINITION. A **coalitional game with transferable payoff** (or characteristic function game) is a pair $\langle N, v\rangle$ where $N = \{1, ..., n\}$ is the set of players and for every subset S of N (called a ***coalition***) $v(S) \in \mathbb{R}$ is the total payoff that is available for division among the members of S (called the ***worth*** of S). We assume that the larger the coalition the higher the payoff (this property is called superadditivity):

$$\text{for all disjoint } S, T \subseteq N, \qquad v(S \cup T) \geq v(S) + v(T)$$

What kind of agreement do we expect individuals to get to? An agreement can be thought of as a list $(x_1, x_2, \ldots, x_n)$ where x_i is the proposed payoff to individual i. Let us try to determine the set of **acceptable agreements** by eliminating those that are unacceptable. First of all, an agreement $(x_1, x_2, \ldots, x_n)$ must be feasible, i.e. it cannot be such that $x_1 + x_2 + \ldots + x_n > v(N)$. Thus,

first necessary condition for acceptability:	$x_1 + x_2 + \ldots + v_n \leq v(N)$ (feasibility condition).

Secondly, an agreement $(x_1, x_2, \ldots, x_n)$ would be unacceptable to individual i if $x_i < v(\{i\})$, because if such an agreement were proposed, individual i would do better by refusing to be part of the agreement and acting by herself [thus guaranteeing herself a payoff of $v(\{i\})$]. Thus

second necessary condition for acceptability:	$x_i \geq v(\{i\})$ for all i (individual rationality condition).

Thirdly, an agreement $(x_1, x_2, \ldots, x_n)$ would also be unacceptable if $x_1 + x_2 + \ldots + x_n < v(N)$, because it would require some potential surplus to be wasted.. Thus

third necessary condition for acceptability:	$x_1 + x_2 + \ldots + x_n \geq v(N)$ (Pareto optimality).

Note that the first and third condition together require $x_1 + x_2 + \ldots + x_n = v(N)$.

EXAMPLE. Consider the following game: $N = \{1,2,3\}$ and

$v(\{1\}) = 100$

$v(\{2\}) = 125$

$v(\{3\}) = 50$

$v(\{1,2\}) = 270$

$v(\{1,3\}) = 375$

$v(\{2,3\}) = 350$

$v(\{1,2,3\}) = 500$

Then the following agreement satisfies the three necessary conditions listed above: $x_1 = 120$, $x_2 = 250$, $x_3 = 130$. Is such an agreement likely to be accepted, if proposed to the three individuals? The answer is No, because individuals 1 and 3 would be better off if they walked out of the negotiations and acted independently of individual 2: 1 and 3 together (and without individual 2) can get 375 and they could split this sum in such a way that they are both better off than in the proposed agreement, e.g. 1 gets 180 and 3 gets 195. Thus we need to add further restrictions.

DEFINITION. A proposed agreement $(x_1, x_2, \dots, x_n)$ is *blocked by coalition S* if there exists a vector $(y_1, y_2, \dots, y_n)$ such that:

(1) $y_i > x_i$ for all $i \in S$, (each member of S is better off under the alternative *y*)

(2) $\sum_{i \in S} y_i \le v(S)$ (alternative *y* is feasible for the coalition S.

Thus the coalition S blocks agreement $(x_1, x_2, \dots, x_n)$ if the members of S can withdraw from the negotiations with the rest of the players and achieve among themselves a better allocation of payoffs. Thus,

fourth necessary condition for acceptability:	there is no coalition that blocks the proposed agreement.

DEFINITION. The **core** is the set of proposed agreements that satisfy the above four conditions.

How do we find the core? The following theorem gives us the answer.

THEOREM. A feasible agreement $(x_1, x_2, \dots, x_n)$ is in the core if and only if

$$\sum_{i \in S} x_i \ge v(S) \quad \text{for all } S \subseteq N \quad (S \ne \emptyset)$$

For the intellectually ambitious here is a simple-to-understand proof.

Proof. Let $(x_1, x_2, \ldots, x_n)$ be a feasible allocation that satisfies the above property. Then taking $S = \{i\}$ we get individual rationality and taking $S = N$ we get Pareto optimality. On the other hand, if there were a coalition S that could block $(x_1, x_2, \ldots, x_n)$ with $(y_1, y_2, \ldots, y_n)$ then we would have that $y_i > x_i$ for all $i \in S$ and $\sum_{i\in S} y_i \le v(S)$. But $y_i > x_i$ for all $i \in S$ implies that $\sum_{i\in S} y_i > \sum_{i\in S} x_i$. By hypothesis $\sum_{i\in S} x_i \ge v(S)$. Thus $\sum_{i\in S} y_i > v(S)$ yielding a contradiction.

Conversely, let $(x_1, x_2, \ldots, x_n)$ be an allocation in the core. We want to show that it must satisfy the property that $\sum_{i\in S} x_i \ge v(S)$ for all $S \subseteq N$ ($S \neq \varnothing$). Suppose not. Then there exists an $S \subseteq N$ such that $\sum_{i\in S} x_i < v(S)$. Let $a = v(S) - \sum_{i\in S} x_i > 0$ and consider the following allocation $(y_1, \ldots, y_n)$:

$$y_i = \begin{cases} x_i + \dfrac{a}{|S|} & \text{if } i \in S \\ y_i = v(\{i\}) & \text{if } i \notin S \end{cases} \qquad \text{(where } |S| \text{ denotes the number of elements in S)}$$

Then $(y_1, \ldots, y_n)$ is feasible (by the superadditivity condition) and S blocks $(x_1, x_2, \ldots, x_n)$ with $(y_1, \ldots, y_n)$, since $y_i > x_i$ for all $i \in S$ and $\sum_{i\in S} y_i = v(S)$. Thus $(x_1, x_2, \ldots, x_n)$ cannot be in the core, yielding a contradiction. ■

EXAMPLE. In the above example, by the theorem the core consists of all the triples (x_1,x_2,x_3) such that:

$x_1 \geq v(\{1\}) = 100$	(1)
$x_2 \geq v(\{2\}) = 125$	(2)
$x_3 \geq v(\{3\}) = 50$	(3)
$x_1 + x_2 \geq v(\{1,2\}) = 270$	(4)
$x_1 + x_3 \geq v(\{1,3\}) = 375$	(5)
$x_2 + x_3 \geq v(\{2,3\}) = 350$	(6)
$x_1 + x_2 + x_3 = v(\{1,2,3\}) = 500$	(7)

From (5) and (7) we get that $x_2 \leq 125$. This, together with (2), gives

$$x_2 = 125. \quad (8)$$

From (7) and (8) we get that $x_1 + x_3 = 375$ so that

$$x_1 = 375 - x_3. \quad (9)$$

From (4) and (8) we get that

$$x_1 \geq 270 - 125 = 145. \quad (10)$$

From (9) and (10) we get that $375 - x_3 \geq 145$ i.e. $x_3 \leq 230$.

From (6) and (8) we get that

$$x_3 \geq 225. \quad (10)$$

Thus the core is the set of triples (x_1, x_2, x_3) such that $x_1 = 375 - x_3$, $x_2 = 125$ and $225 \leq x_3 \leq 230$.