Definition. Contract ***C*** **is *Pareto dominated* by contract *B*** if: or B Pareto dominates C

either $\begin{cases} E[U_A(B)] > E[U_A(C)] \\ \text{and} \\ E[U_P(B)] \geq E[U_P(C)] \end{cases}$ that is, $B \succ_A C$ and $B \succsim_P C$

or $\begin{cases} E[U_P(B)] > E[U_P(C)] \\ \text{and} \\ E[U_A(B)] \geq E[U_A(C)] \end{cases}$ that is, $B \succ_P C$ $B \succsim_A C$

Definition. A contract that is not Pareto dominated by any other contract is called ***Pareto efficient*** **(or Pareto optimal)**. Thus contract C is Pareto efficient if for every other contract D, either

$$E[U_P(D)] < E[U_P(C)]$$

or

$$E[U_A(D)] < E[U_A(C)]$$

or both.

$$\textbf{Example.}\ X^{G} = 1{,}000,\quad X^{B} = 600,\quad \overset{\nearrow \text{prob of } G}{p = \frac{1}{3}} \quad U_P(m) = \sqrt{m} \quad \text{and} \quad \underbrace{U_A(m) = m}.$$

$$C = (\underset{w_G}{400}, \underset{w_B}{400}) \text{ is Pareto dominated by contract } B = (\underset{w_G}{676}, \underset{w_B}{276}):$$

$$\mathbb{E}\left[U_P(B)\right] = \frac{1}{3}\sqrt{1000-676} + \frac{2}{3}\sqrt{600-276} = \sqrt{324} = 18 \checkmark$$

$$\mathbb{E}\left[U_P(C)\right] = \frac{1}{3}\sqrt{1{,}000-400} + \frac{2}{3}\sqrt{600-400} = 17.6$$

$B \succ_P C$

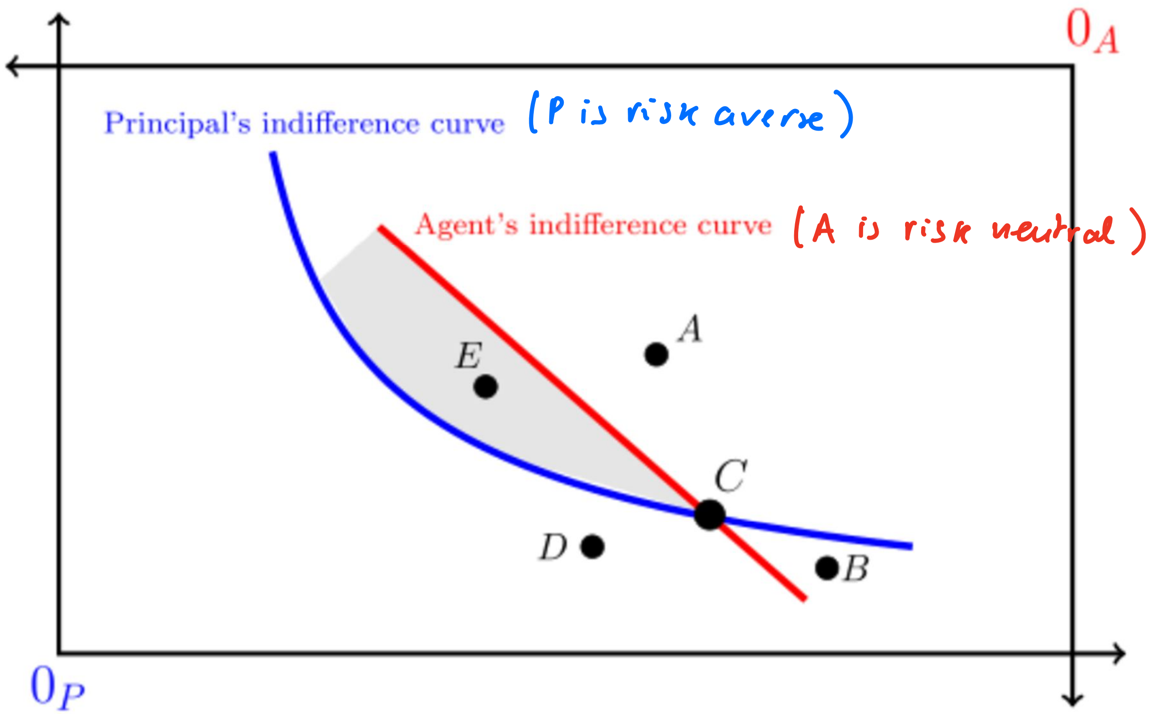
and

$$\mathbb{E}\left[U_A(B)\right] = \frac{1}{3}676 + \frac{2}{3}276 = \underbrace{409.33}_{\checkmark}$$

$B \succ_A C$

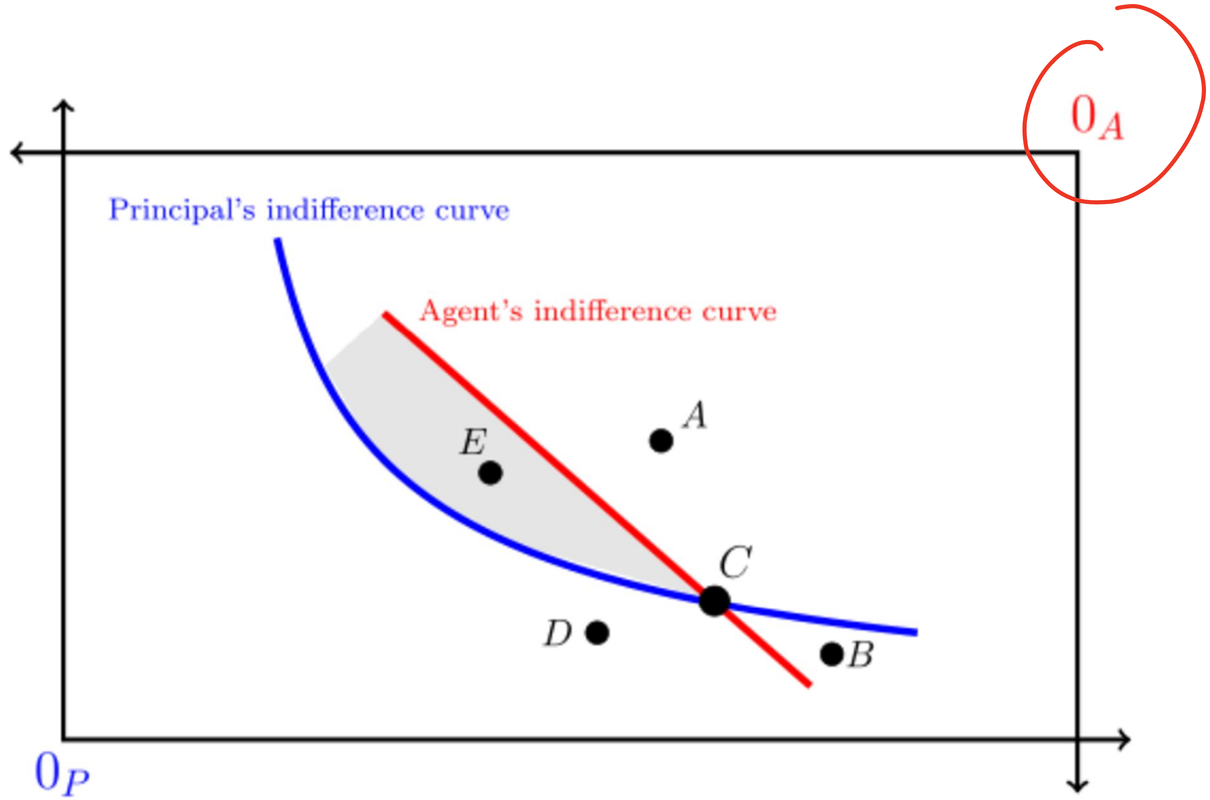
$$\mathbb{E}\left[U_A(C)\right] = 400 = \left(\frac{1}{3}400 + \frac{2}{3}400\right)$$

B Pareto dominates C



For the Principal: $C \succ_P D$ $\quad E \succ_P C$ $\quad A \succ_P C$

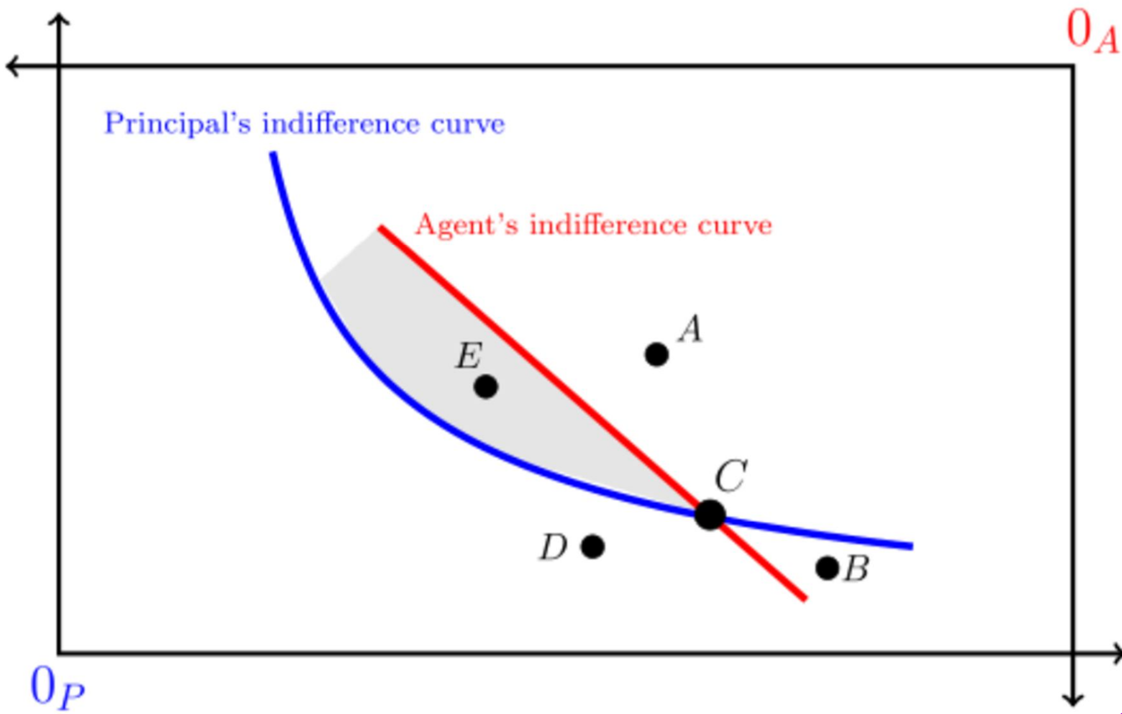
$C \succ_P B$



For the Agent:

$C \succ_A A$ $\quad E \succ_A C$

$C \succ_A B$ $\quad D \succ_A C$



C is Pareto dominated by E so C is not Pareto efficient

For the Principal: $C \succ_P D$ $\quad C \succ_P B$ $\quad \boxed{E \succ_P C}$ $\quad A \succ_P C$ $\quad A \succ_P E$

For the Agent: $C \succ_A A$ $\quad B \succ_A A$ $\quad C \succ_A B$ $\quad \boxed{E \succ_A C}$ $\quad D \succ_A C$

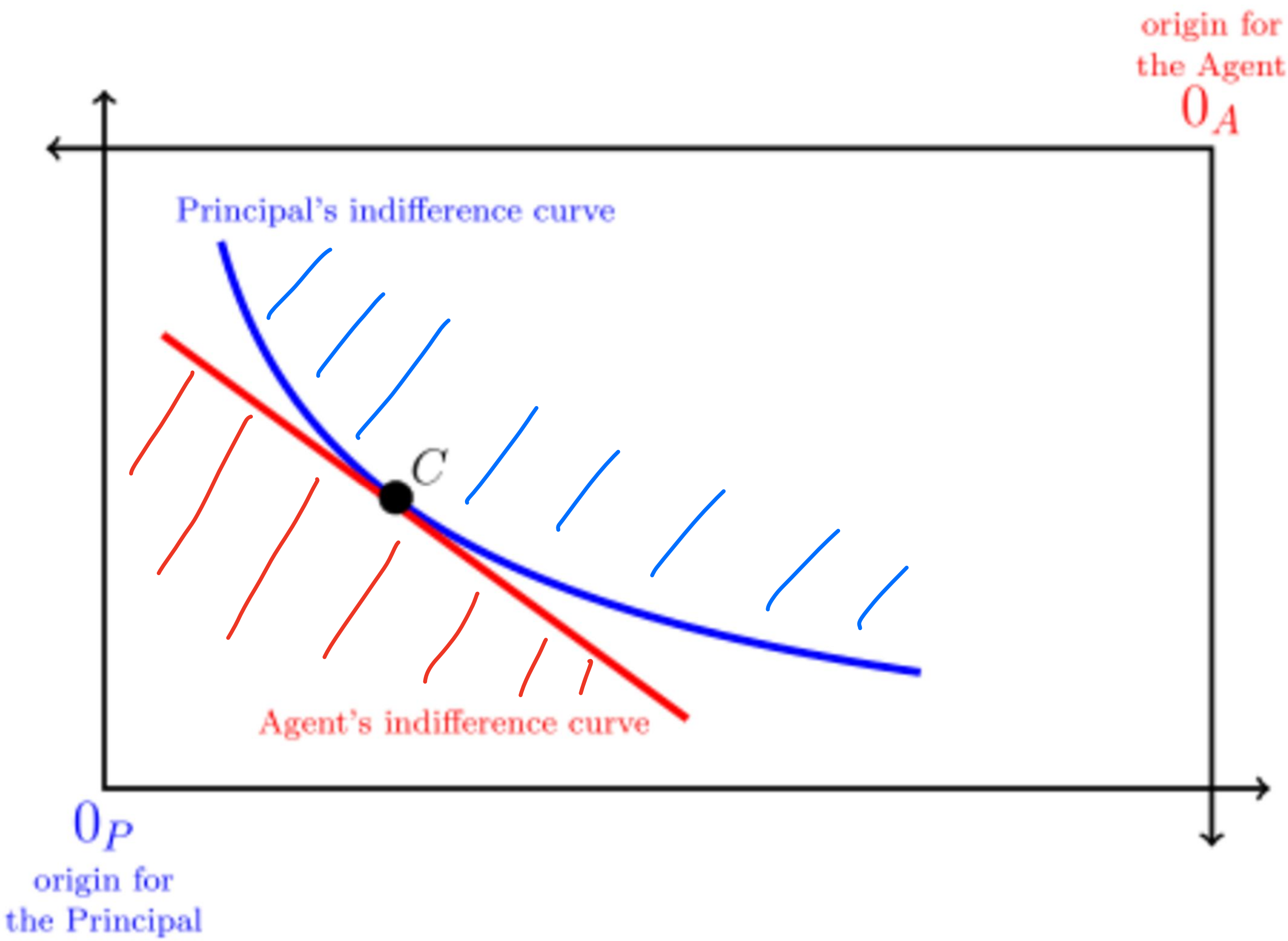
$D \succ_A E$

Thus C is Pareto dominated by E (or E Pareto dominates C). So C is not Pareto efficient.

Any contract C at which the indifference curves cross cannot be Pareto efficient, because any contract in the area between the two curves is Pareto superior to (or Pareto dominates) C.

Thus a contract C **in the interior of the box** is Pareto efficient if and only if the two indifference curves (of Principal and Agent) are **tangent** at C.

Example:



Pareto efficient risk sharing

We saw that a contract $C=\left(w_C^G, w_C^B\right)$ in the **interior** of the Edgeworth box ($0<w^G<X^G$ and $0<w^B<X^B$) is Pareto efficient if and only if the two indifference curves through C are tangent at that point.

- Slope of Principal's indifference curve at $C=\left(w_C^G, w_C^B\right)$: $-\dfrac{p}{1-p}\dfrac{U_P'(X^G-w_C^G)}{U_P'(X^B-w_C^B)}$

- Slope of Agent's indifference curve at $C=\left(w_C^G, w_C^B\right)$: $-\dfrac{p}{1-p}\dfrac{U_A'(w_C^G)}{U_A'(w_C^B)}$

Thus the two are equal if and only if

$$\frac{U_P'(X^G-w_C^G)}{U_P'(X^B-w_C^B)} = \frac{U_A'(w_C^G)}{U_A'(w_C^B)}$$

$U_A(\$m) = am + b \qquad a > 0$

Case 1: Principal risk averse, Agent risk neutral

Agent's utility function can be taken to be $U_A(\$m) = m \qquad U_A'(\$m) = 1$

$$\frac{U_A'(w_C^G)}{U_A'(w_C^B)} = \frac{1}{1} = 1$$

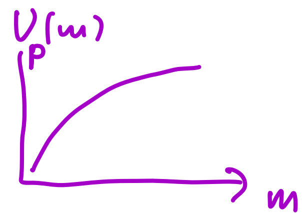
Hence the required equality $\dfrac{U_P'(X^G - w_C^G)}{U_P'(X^B - w_C^B)} = \underbrace{\dfrac{U_A'(w_C^G)}{U_A'(w_C^B)}}_{1}$ reduces to $U_P'(X^G - w_C^G) = U_P'(X^B - w_C^B)$

$$\Rightarrow X^G - w_C^G = X^B - w_C^B \qquad (*)$$

$\Downarrow$

C lies on 45° line for Principal

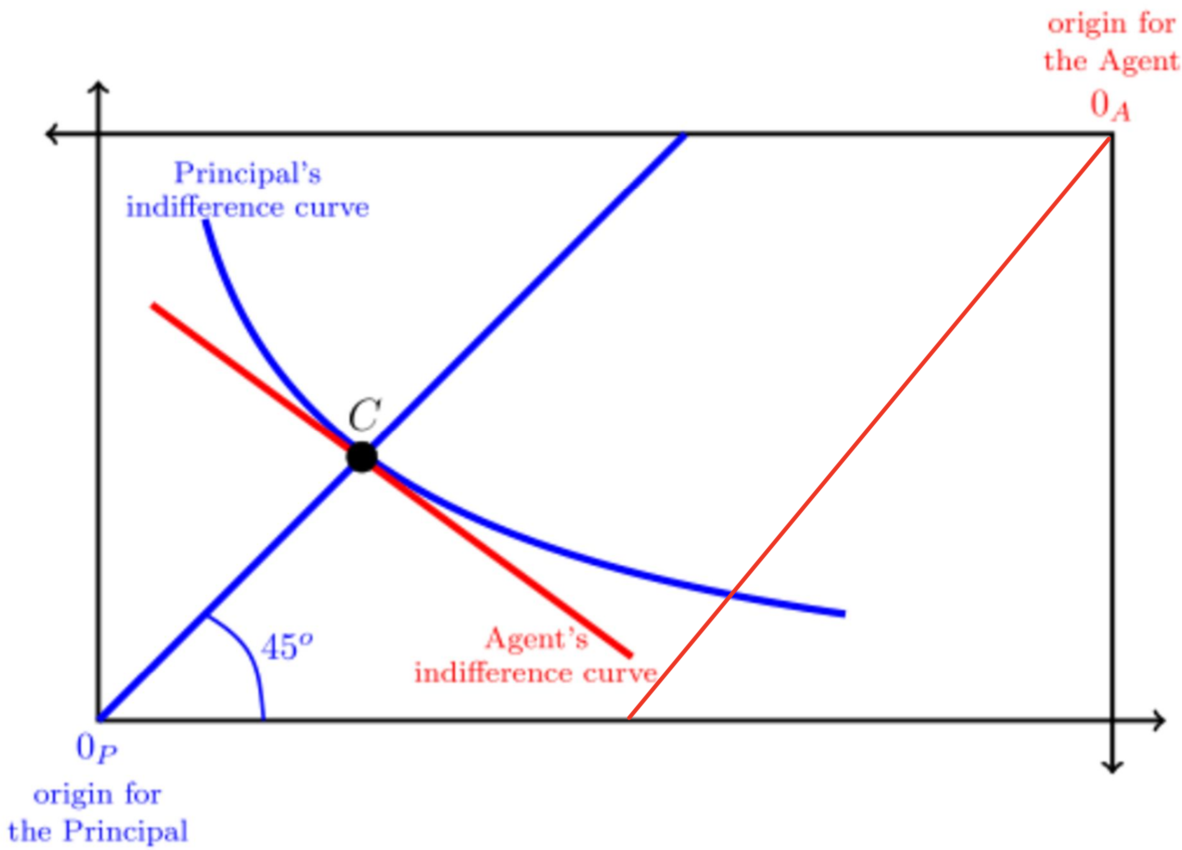
Since Principal is risk-averse



Since U_P is strictly concave,

Thus (*) is satisfied if and only if

That is, contract C must be on the 45° line out of the origin for the Principal:



Case 2: Principal risk neutral, Agent risk averse

Principal's utility function can be taken to be $U_p(\$m) = m$ $U_p'(\$m) = 1$

Hence the required equality $\overbrace{\frac{U_P'(X^G - w_C^G)}{U_P'(X^B - w_C^B)}}^{1 = \frac{1}{1}} = \frac{U_A'(w_C^G)}{U_A'(w_C^B)}$ reduces to $U_A'(w_c^G) = U_A'(w_c^B)$

$$\Downarrow \qquad (*)$$

$$w_c^G = w_c^B$$

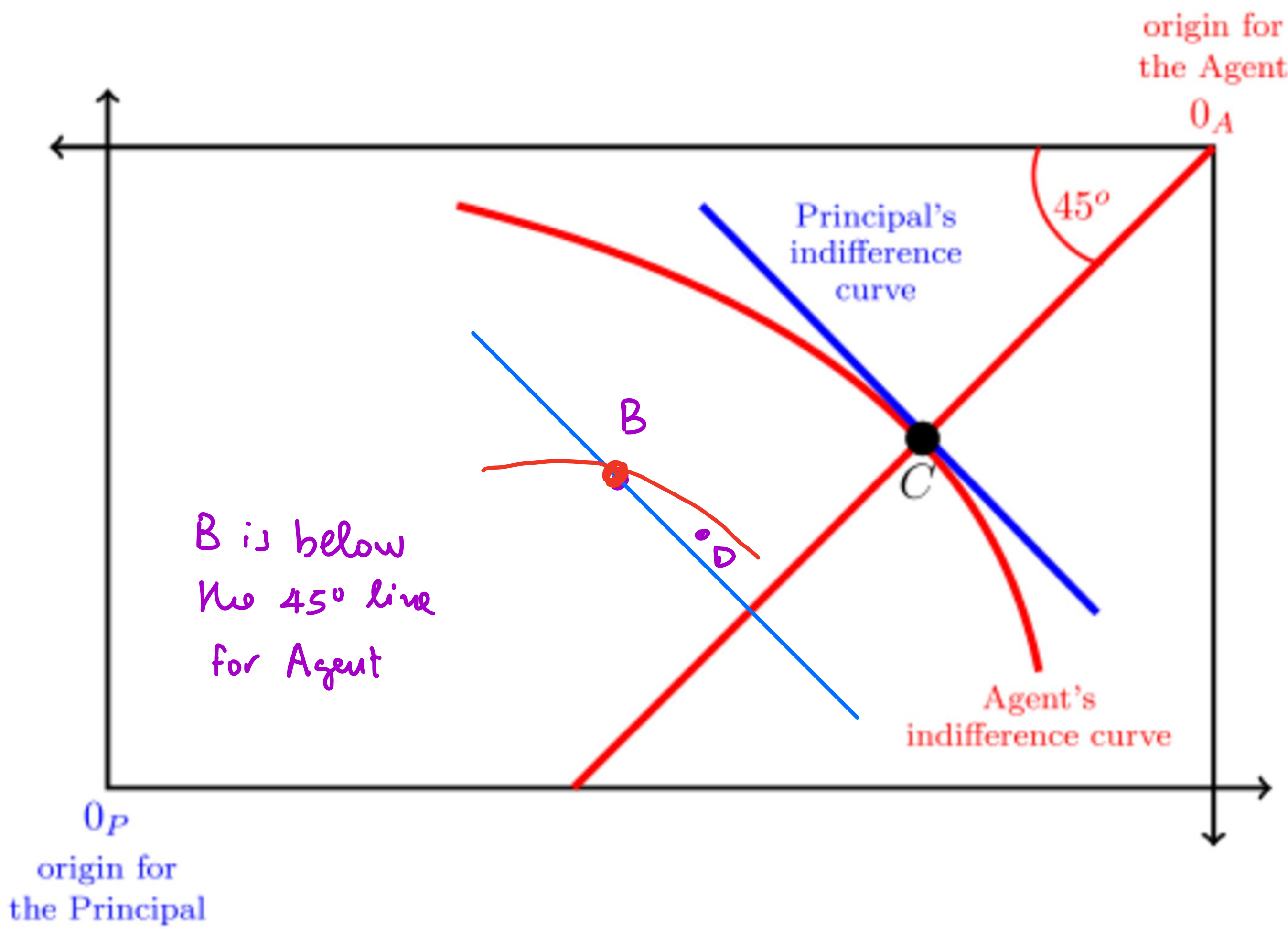
Since U_A is strictly concave,

$$\Downarrow$$

C lies on 45° line for the Agent

Thus (*) is satisfied if and only if

Thus contract C must be on the 45° line out of the origin for the Agent:



General principle: when one party is risk averse and the other is risk neutral, the risk-neutral party must bear all the risk (that is, the risk-averse party must be guaranteed a fixed level of wealth).

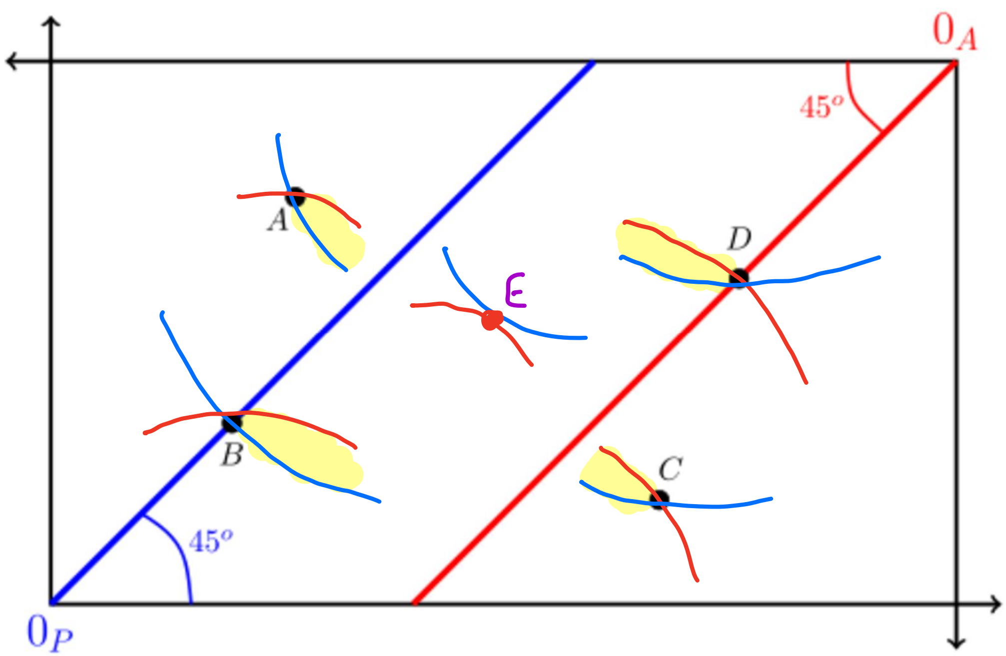
Case 3: both Principal and Agent risk averse

Recall from Week 4 (04A) that (IC = indifference curve)

- at a point **above** the 45° line, **slope of IC is, in absolute value, greater than** $\frac{p}{1-p}$
- at a point **on** the 45° line, **slope of IC is, in absolute value, equal to** $\frac{p}{1-p}$
- at a point **below** the 45° line, **slope of IC is, in absolute value, less than** $\frac{p}{1-p}$

Slope of P's ind curve at B = $\frac{p}{1-p}$ in absolute value

B is below A's 45° line so slope at B is < $\frac{p}{1-p}$



P-slope = absolute value of slope of Principal's indifference curve

A-slope = absolute value of slope of Agent's indifference curve

- At point *A*,
- At point *B*,
- At point *D*,
- At point *C*,

Example. $U_P(m) = \sqrt{m}$ and $U_A(m) = 82 - \left(10 - \frac{m}{100}\right)^2 - 1$. Let $X^G = 800$ and $X^B = 200$. Consider the contract **($w^G = 400$, $w^B = 100$)**. Is it Pareto efficient? We have to check if equality of the slopes holds.

Agent gets 50% of the outcome

$$U'_P(m) = \frac{1}{2\sqrt{m}}$$

$$U'_A(m) = \frac{1{,}000 - m}{5{,}000}$$

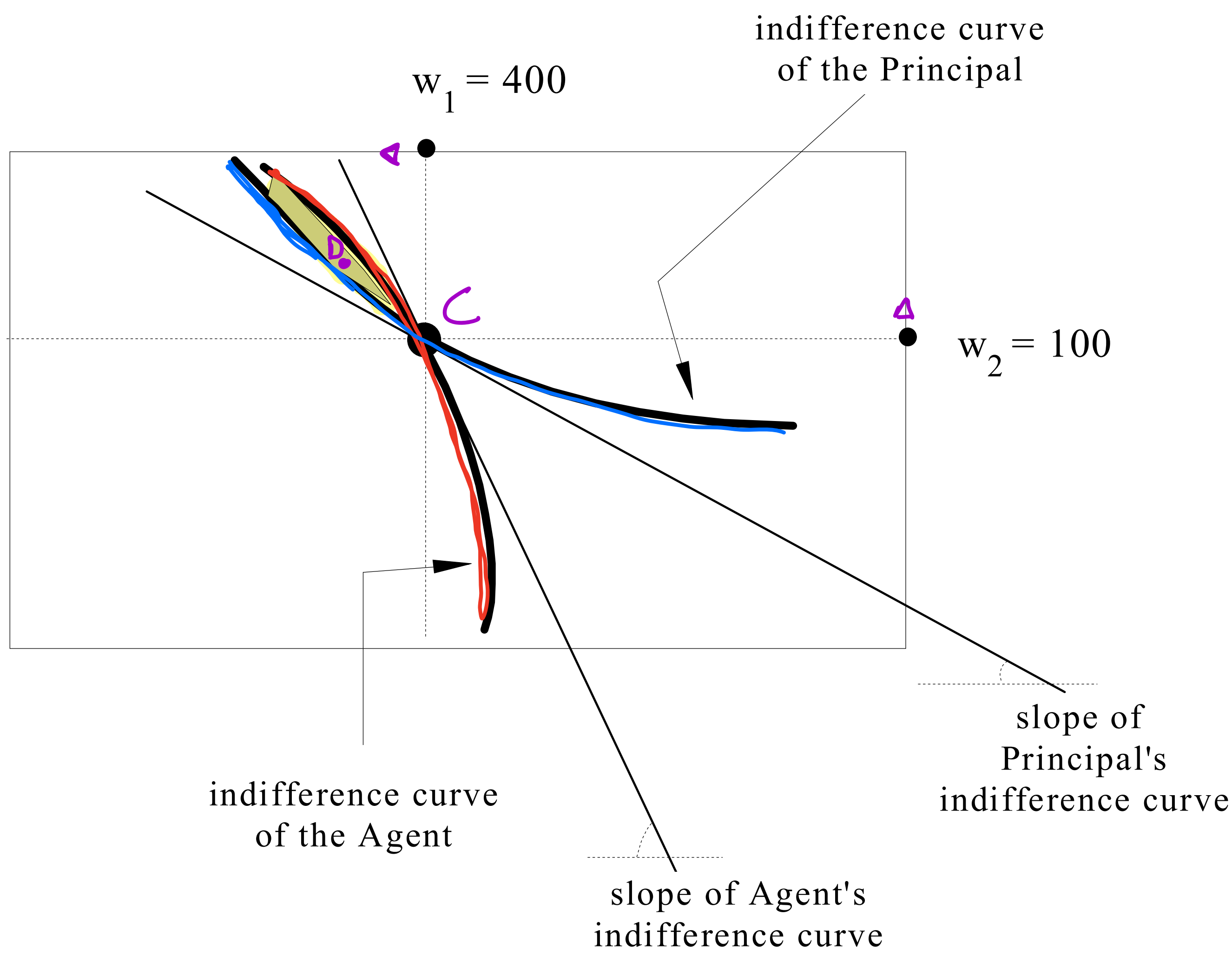
Principal: $$\frac{U'_P(X^G - w^G)}{U'_P(X^B - w^B)} = \frac{\frac{1}{2\sqrt{400}}}{\frac{1}{2\sqrt{100}}} = \frac{1}{2}$$

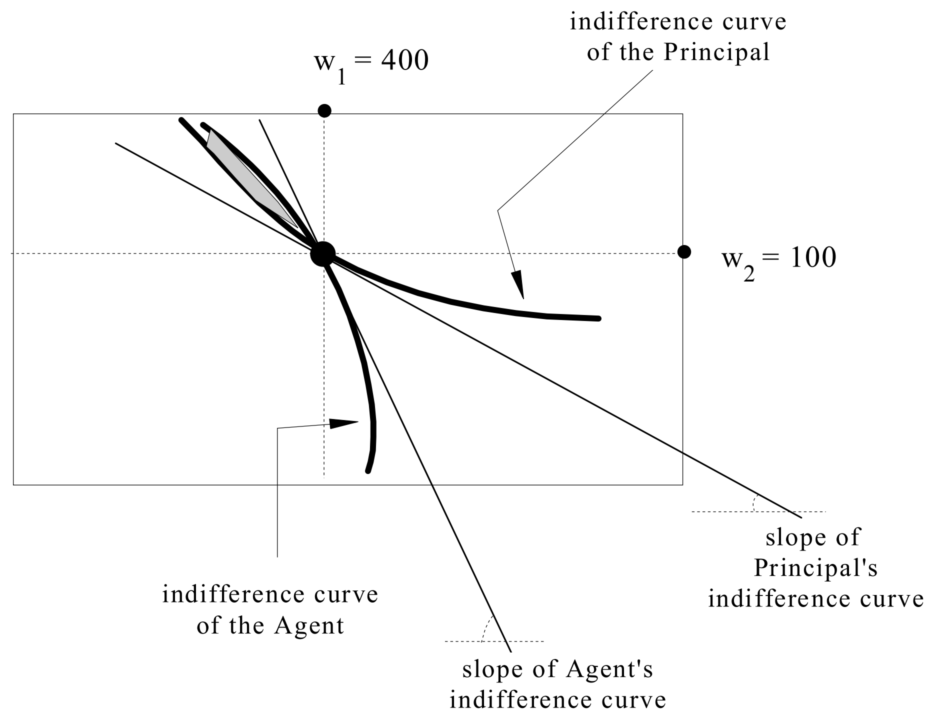
$\neq$

Agent: $$\frac{\frac{1{,}000-400}{5000}}{\frac{1{,}000-100}{5000}} = \frac{2}{3}$$

Contract not Pareto efficient

The indifference curve of the Principal is less steep than the indifference curve of the Agent at point (w^G= 400, w^B= 100): see the following figure



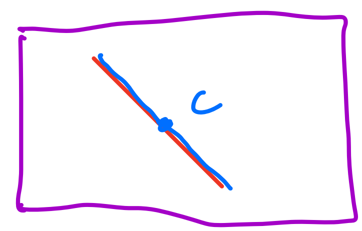


Let S = ($w^G = 400$, $w^B = 100$) be the contract under consideration and let T = ($w^B = 401$, $w^B = 99.7$). Then

$$\mathbb{E}\left[U_P(S)\right] = \qquad\qquad \mathbb{E}\left[U_P(T)\right] =$$

$$\mathbb{E}\left[U_A(S)\right] = \qquad\qquad \mathbb{E}\left[U_A(T)\right] =$$

Case 4: both Principal and Agent risk neutral



every contract
is Pareto efficient