Measuring risk aversion

How to identify risk aversion: $U''(x) < 0$

Can there be more or less risk aversion?

Even the same utility function, **the degree of risk aversion of an individual varies with her level of wealth.**

$U(x) = \sqrt{x}$. Initial wealth: W_0.

What is the **risk premium** associated with this lottery? I**t depends on** W_0**.**

Suppose that $W_0 = 50$

Suppose that Suppose that $W_0 = 1,000$

Thus she is less risk averse when her wealth is $1,000 than when her wealth is $50.

We compared two related lotteries **given some fixed preferences (i.e. a fixed utility function).**

Now **fix a lottery *L* and consider different preferences (that is, different utility functions)**.

Take the risk premium of the lottery as a measure of the intensity of risk aversion.

Initial wealth: 50. Wealth lottery: $L = \begin{pmatrix} 0 & 100 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\mathbb{E}[L] = 50$

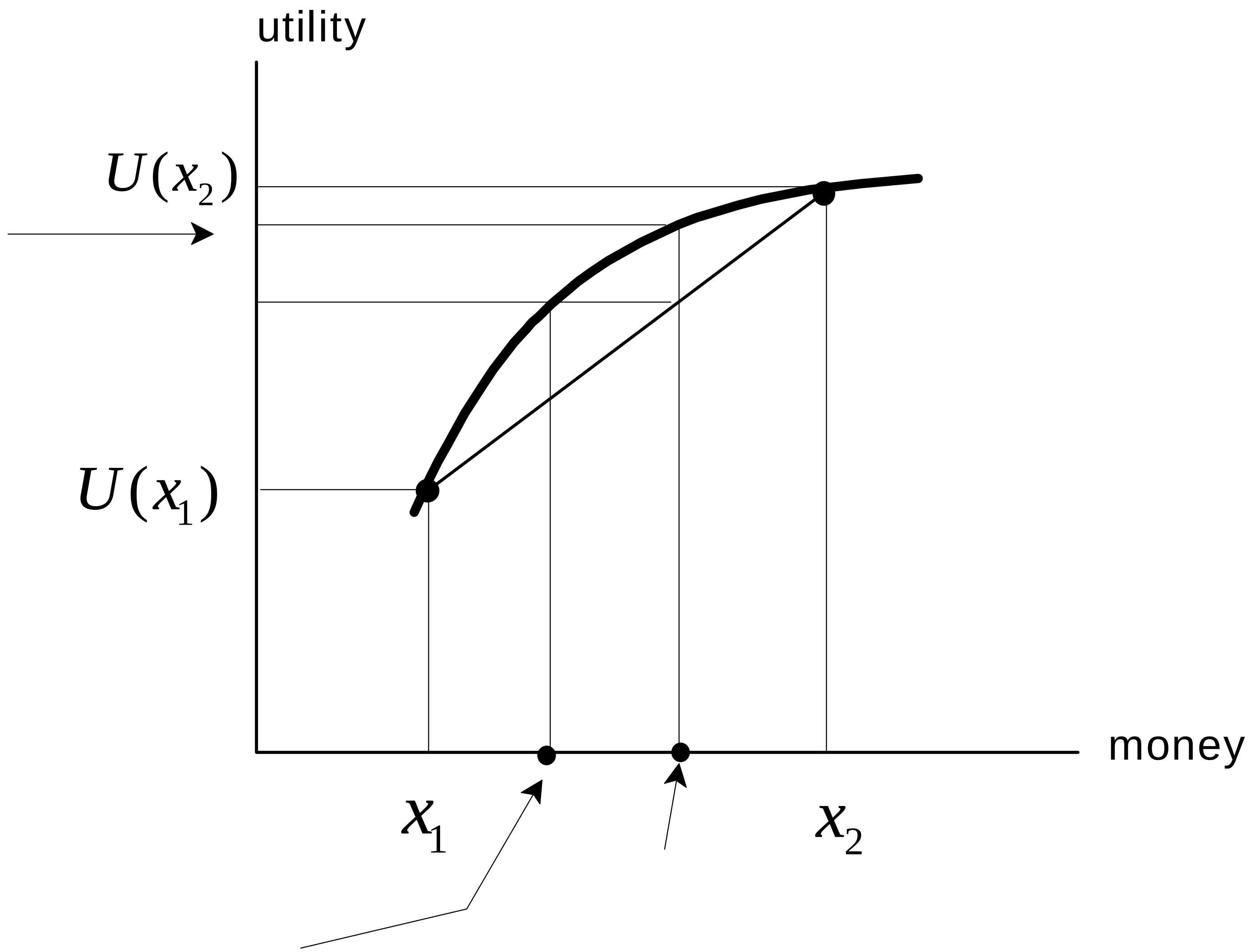
- $\boxed{U(x) = \sqrt{x}}$ then, as we saw before, the risk premium is the solution to

  $\sqrt{50 - R} = \underbrace{5}_{= \mathbb{E}[U(L)]}$ which is $\boxed{R = \$25}$

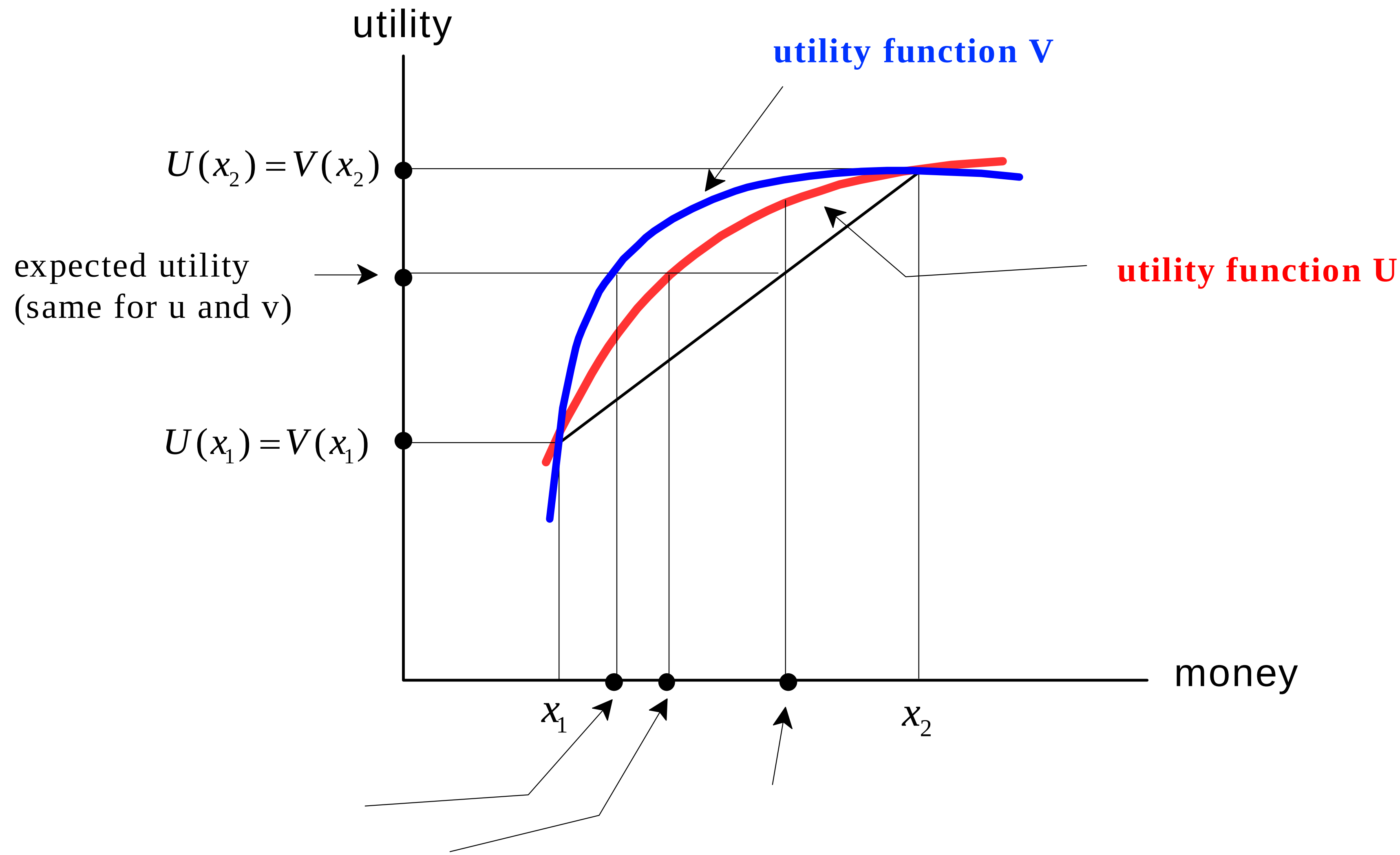
- If her utility function is $\boxed{U(x) = \ln(x+1)}$

Thus the utility function ln(x+1) embodies more risk aversion then the function $\sqrt{x}$ relative to lottery $\begin{pmatrix} 0 & 100 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. But perhaps there is another lottery relative to which the function $\sqrt{x}$ displays more (or the same) risk aversion than the utility function ln(x +1)?

Graphical representation of the risk premium:



A more concave utility function is associated with a larger risk premium for the same lottery:



(2) Check that the risk premium is a meaningful measure, that is, that it is invariant to an allowed transformation of the utility function.

$$L = \begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}, \quad \mathbb{E}[L] = p_1 x_1 + p_2 x_2 + \dots + p_n x_n$$

Utility function $U(\$x)$. $\mathbb{E}[U(L)] =$

R_{UL} solution to

Now let $V(x) = aU(x) + b$ with $a > 0$

R_{VL} solution to

$V\left(\mathbb{E}[L] - R\right) = \mathbb{E}[V(L)]$ if and only if $U\left(\mathbb{E}[L] - R\right) = \mathbb{E}[U(L)]$. Hence $R_{VL} = R_{UL}$

Definition. Utility function U embodies more risk aversion that utility function V if

$R_{UL} > R_{VL}$ for every non-degenerate money lottery L.

Short of trying every possible lottery, is there a way to determine if U embodies more risk aversion than V?

Arrow-Pratt measure of risk aversion:

First, let us verify that it is a meaningful measure, that is, that it is invariant to an allowed transformation of the utility function

Let $V(x) = aU(x) + b$ for every $x \geq 0$ with $a > 0$. $V'(x) =$ and $V''(x) =$

Examples.

$$\boxed{U(x)=\sqrt{x}}=x^{\frac{1}{2}}$$

$$\boxed{U(x)=\ln(x)}$$

Note that both display decreasing risk aversion as *x* increases

Theorem. Let U(x) and V(x) be two strictly concave functions. Then the following conditions are equivalent:

1. $R_{VL} > R_{UL}$ for every non-degenerate wealth lottery *L*
2. $A_V(x) > A_U(x)$ for every $x > 0$.

Ranking lotteries

Given two money lotteries L and M when would any two individuals agree that L is better than M, no matter their attitude to risk? Assume throughout that every individual prefers more money to less, that is, that each individual's utility function is strictly increasing.

Everybody will agree that   is better than   .

What about   and   ?

$\mathbb{E}[L]$ and $\mathbb{E}[M] =$

For a risk-neutral person:

For a risk-averse person with utility function $U(x) = \sqrt{x}$

$\mathbb{E}[U(L)] =$ $\mathbb{E}[U(M)] =$

However, there are lotteries that can be unambiguously ranked in the sense that everybody ranks them the same way.

$$L = \begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \qquad M = \begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}.$$

Note that the **basic outcomes are the same** in both lotteries and for this part assume that the **prizes are listed in increasing order**: $0 \le x_1 < x_2 < \dots < x_n$.

Define the **cumulative distribution function** (cdf) for lottery L as follows:

$P_i = p_1 + \dots + p_i$ for every $i = 1,\dots,n$:

$$L = \begin{pmatrix} \$x_1 & \$x_2 & \$x_3 & \dots & \$x_n \\ p_1 & p_2 & p_3 & \dots & p_n \\ & & & & \end{pmatrix}$$
$$P:$$

P_i is the probability that $x \le x_i$.

define the cumulative probability distribution for lottery M as follows: $Q_i = q_1 + \ldots + q_i$ for every $i = 1,\ldots,n$:

$$M = \begin{pmatrix} \$x_1 & \$x_2 & \$x_3 & \ldots & \$x_n \\ q_1 & q_2 & q_3 & \ldots & q_n \\ Q: & & & & \end{pmatrix}$$

Definition. We say that L **first-order stochastically dominates** M and write $L >_{FSD} M$ if $P_i \le Q_i$ for ever $i = 1, 2, \ldots, n,$ with at least one strict inequality.

Example 1.

$$L = \begin{pmatrix} \$40 \\ 1 \end{pmatrix} \text{ and } M = \begin{pmatrix} \$20 & \$60 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Example 2.

$$L = \begin{pmatrix} \$20 & \$40 & \$50 & \$60 \\ \frac{1}{12} & \frac{3}{12} & \frac{6}{12} & \frac{2}{12} \end{pmatrix} \text{ and } \mathrm{M} = \begin{pmatrix} \$20 & \$40 & \$50 & \$60 \\ \frac{1}{12} & \frac{4}{12} & \frac{5}{12} & \frac{2}{12} \end{pmatrix}.$$

Theorem. $L >_{FSD} M$ if and only if $\mathbb{E}[U(L)] > \mathbb{E}[U(M)]$ for every strictly increasing utility function *U*.

Thus if lottery *L* first-order stochastically dominates lottery *M* then it is unambiguously better than *M*, in the sense that everybody, no matter what their attitude to risk, prefers *L* to *M*.

Now **focus on risk-averse individuals** and ask when any two risk-averse individuals would agree that a lottery *M* is worse than another lottery *L*, in which case we can interpret this as ***M* being more risky than *L***.

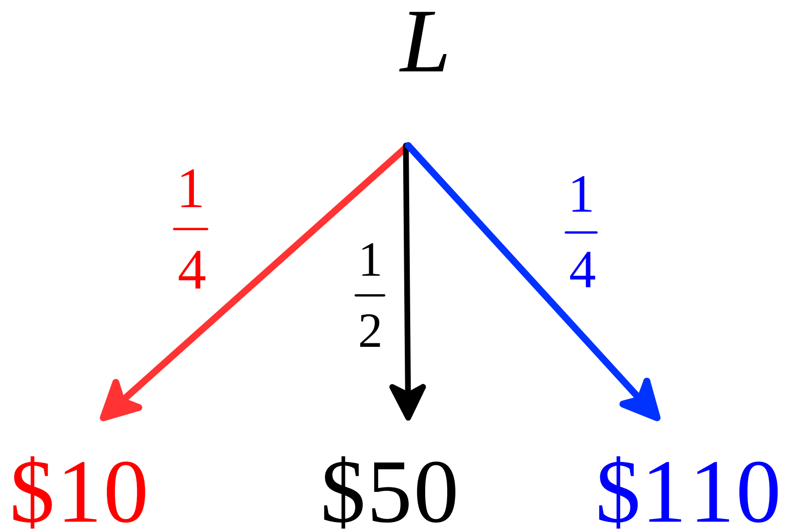
To begin with the two lotteries ought to be similar: $\mathbb{E}[L] = \mathbb{E}[M]$, in which case a risk-neutral individual would be indifferent between the two. Hence if a risk-averse person is not indifferent it must be because one is "more risky" than the other.

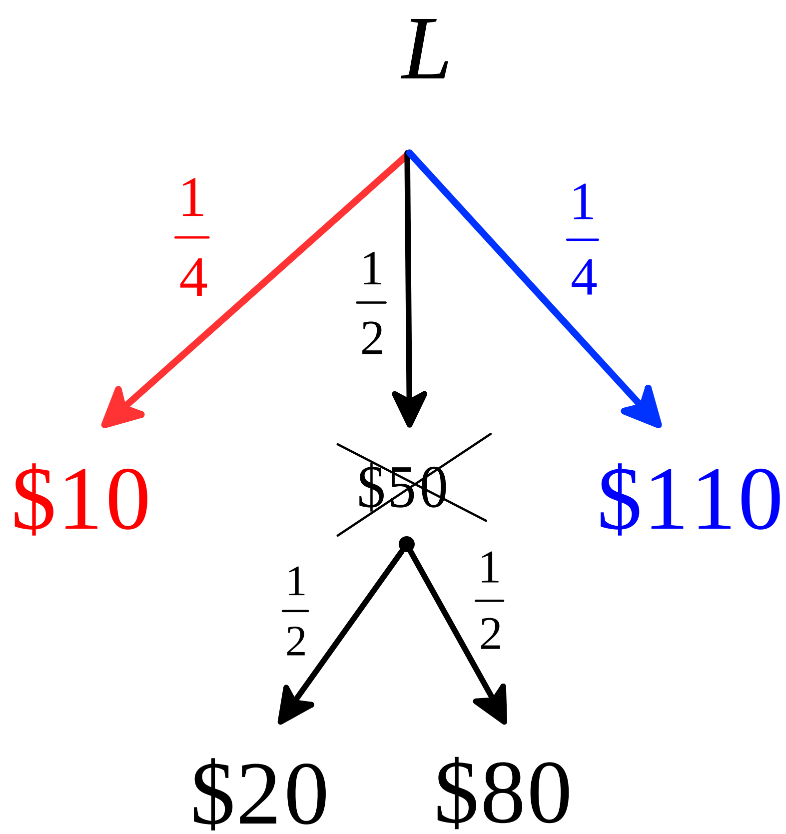
$$L = \begin{pmatrix} \$50 \\ 1 \end{pmatrix}$$

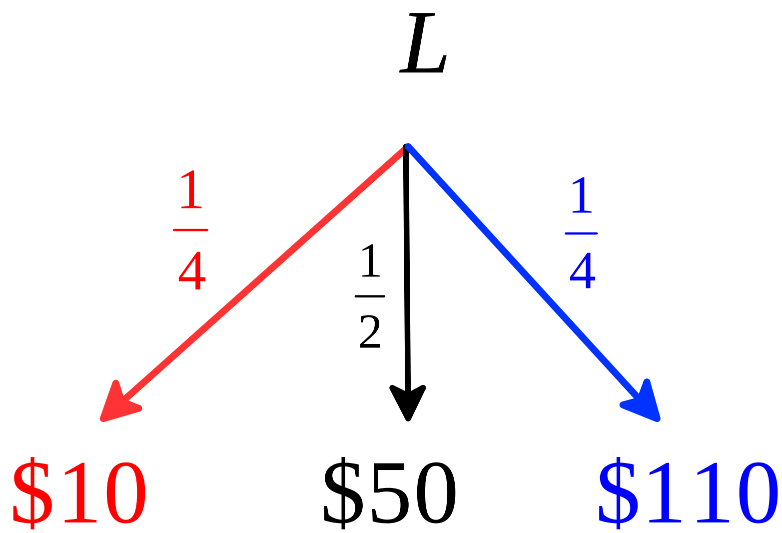
$$L = \begin{pmatrix} \$10 & \$50 & \$110 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \text{ with } \quad \mathbb{E}[L] = 55$$

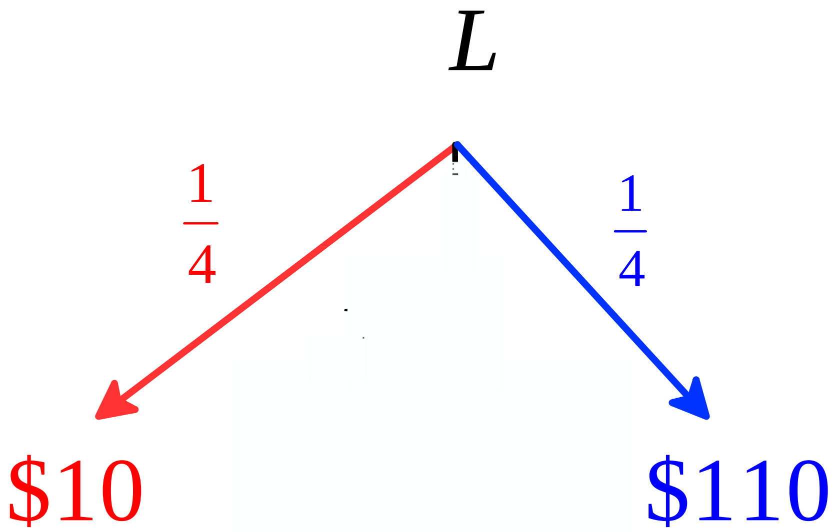
$$\begin{matrix} \$50 \\ \frac{1}{2} \end{matrix}$$

$$L = \begin{pmatrix} \$10 & \$50 & \$110 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \text{ with } \quad \mathbb{E}[L] = 55$$






L
1/4
1/2
1/4
$10
$50
$110


L
1/4
1/4
$10
$110

Write $L >_{SSD} M$ to mean that ***L* dominates *M*** in the sense of **second-order stochastic dominance.**

Definition. $L >_{SSD} M$ if M can be obtained from L by a finite sequence of mean - preserving spreads, that is, if there is a sequence of money lotteries $\langle L_1, L_2, \ldots, L_m \rangle$ (with $m \geq 2$) such that:

$$(1)\ L_1 = L,$$
$$(2)\ L_m = M$$
$$(3)\ \text{for every } i = 1, \ldots, m-1,\ L_i \rightarrow_{MPS} L_{i+1}$$

Theorem. $L >_{SSD} M$ if and only if $\mathbb{E}[U(L)] > \mathbb{E}[U(M)]$ for every strictly increasing and strictly concave utility function *U*.

We don't have to reduce the probability to zero: $L = \begin{pmatrix} \$10 & \$50 & \$110 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

Take away some of the probability of \$50, say $\frac{3}{10}$ and spread it between a lower amount, say \$15, and a higher amount, say \$90:

$$M = \begin{pmatrix} \$10 & \$15 & \$50 & \$90 & \$110 \\ & & & & \end{pmatrix}$$

For this to be a mean preserving spread we need

$$M = \begin{pmatrix} \$10 & \$15 & \$50 & \$90 & \$110 \\ & & & & \end{pmatrix}$$