How to construct or discover a person's von Neumann-Morgenstern utility function

$$Z = \{z_1, z_2, z_3, z_4\}$$

Question 1: how do you rank the basic outcomes?

Suppose her ranking is

		normalized utility
best	z_3	1
	z_1, z_4	$\frac{3}{4}$
worst	z_2	0

Question 2: what value of *p* would make you indifferent between the following two lotteries?

$$\begin{pmatrix} z_1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{array}{cc} \text{best} & \text{worst} \\ \end{array} \begin{pmatrix} z_3 & z_2 \\ p & 1-p \end{pmatrix}$$

Suppose she says $p = \frac{3}{4}$

$$\begin{pmatrix} z_1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_3 & z_2 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

$$EU = 1 \cdot U(z_1) = U(z_1) \quad = \quad \frac{3}{4} U(z_3) + \frac{1}{4} U(z_2) = \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 0 = \frac{3}{4}$$

		UTILITY
best	z_3	100
	z_1, z_4	77.5
worst	z_2	10

normalize this:

1. subtract 10	90
	67.5
	0
2. divide by 90	1
	$\frac{3}{4}$
	0

$$p = \frac{3}{4}$$

$$\begin{pmatrix} z_1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_3 & z_2 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

$$U(z_1) = \frac{3}{4} U(z_3) + \frac{1}{4} U(z_2) = \frac{3}{4} 100 + \frac{1}{4} 10 = 75 + 2.5 = 77.5$$

$Z = \{z_1, z_2, z_3, z_4, z_5\}$

Q. 1 : what is the ranking?

		u
best	z_2	1
	z_3	
	z_1, z_4	
worst	z_5	0

Q. 2 what p is such that $\begin{pmatrix} z_3 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_2 & z_5 \\ p & 1-p \end{pmatrix}$

Suppose answer is $p = \frac{3}{5}$

Q. 3 what p is such that
$$\begin{pmatrix} z_1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_2 & z_5 \\ p & 1-p \end{pmatrix}$$

Suppose answer is : $p = \frac{1}{5}$

vNM

Utility function:

z_1	z_2	z_3	z_4	z_5
$\frac{1}{5}$	1	$\frac{3}{5}$	$\frac{1}{5}$	0

$L = \begin{pmatrix} z_1 & z_3 & z_5 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ $M = \begin{pmatrix} z_2 & z_4 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$E[U(L)] = \frac{1}{3}\frac{1}{5} + \frac{1}{3}\frac{3}{5} + \frac{1}{3}0$

$= \frac{1}{15} + \frac{3}{15} = \frac{4}{15}$

$E[U(M)] =$

$\frac{1}{2}1 + \frac{1}{2}\frac{1}{5} =$

$1 + \frac{1}{10} = \frac{11}{10}$

Risk neutral

U identity function

best	\$ 120	120	-30	90		$1 = \frac{90}{90}$
	\$ 60	60	normalize	30	$\times \frac{1}{90}$	$\frac{1}{3} = \frac{30}{90}$
worst	\$ 30	30		0		$0 = \frac{0}{90}$

$$Z = \{ z_1, \ldots, z_n \}$$

Maximum number of questions?

$$n - 2 + 1 = n - 1$$

Restating attitudes to risk in terms of utility

$$L = \begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$$ money lottery

U utility-of-money function.

$$\mathbb{E}[L] = \underbrace{p_1 x_1 + p_2 x_2 + \dots + p_n x_n}_{\text{expected value of L}} \qquad \mathbb{E}[U(L)] = \overbrace{\underbrace{p_1 U(x_1) + p_2 U(x_2) + \dots + p_n U(x_n)}_{\text{expected utility of L}}}^{E[U(L)]}$$

$E[L]$

- Risk averse if $U(E[L]) > E[U(L)]$
- Risk neutral if $U(E[L]) = E[U(L)]$
- Risk loving if $U(E[L]) < E[U(L)]$

Example: $L = \begin{pmatrix} \$16 & \$36 & \$64 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$. Then $\mathbb{E}[L]) = \frac{1}{4}16 + \frac{1}{2}36 + \frac{1}{4}64 =$

$4 + 18 + 16 = 38$

- $U(\$x) = \sqrt{x}$. Then RISK AVERSE

$U(38) = \sqrt{38} = 6.1644$ compare to

$E[U(L)] = \frac{1}{4}\sqrt{16} + \frac{1}{2}\sqrt{36} + \frac{1}{4}\sqrt{64} =$

$= \frac{1}{4}4 + \frac{1}{2}6 + \frac{1}{4}8 = 1 + 3 + 2 = 6$

- $U(\$x) = \frac{x}{2}$. Then RISK NEUTRAL

$U(38) = \frac{38}{2} = 19$ $E[U(L)] = \frac{1}{4}\frac{16}{2} + \frac{1}{2}\frac{36}{2} + \frac{1}{4}\frac{64}{2} = 19$

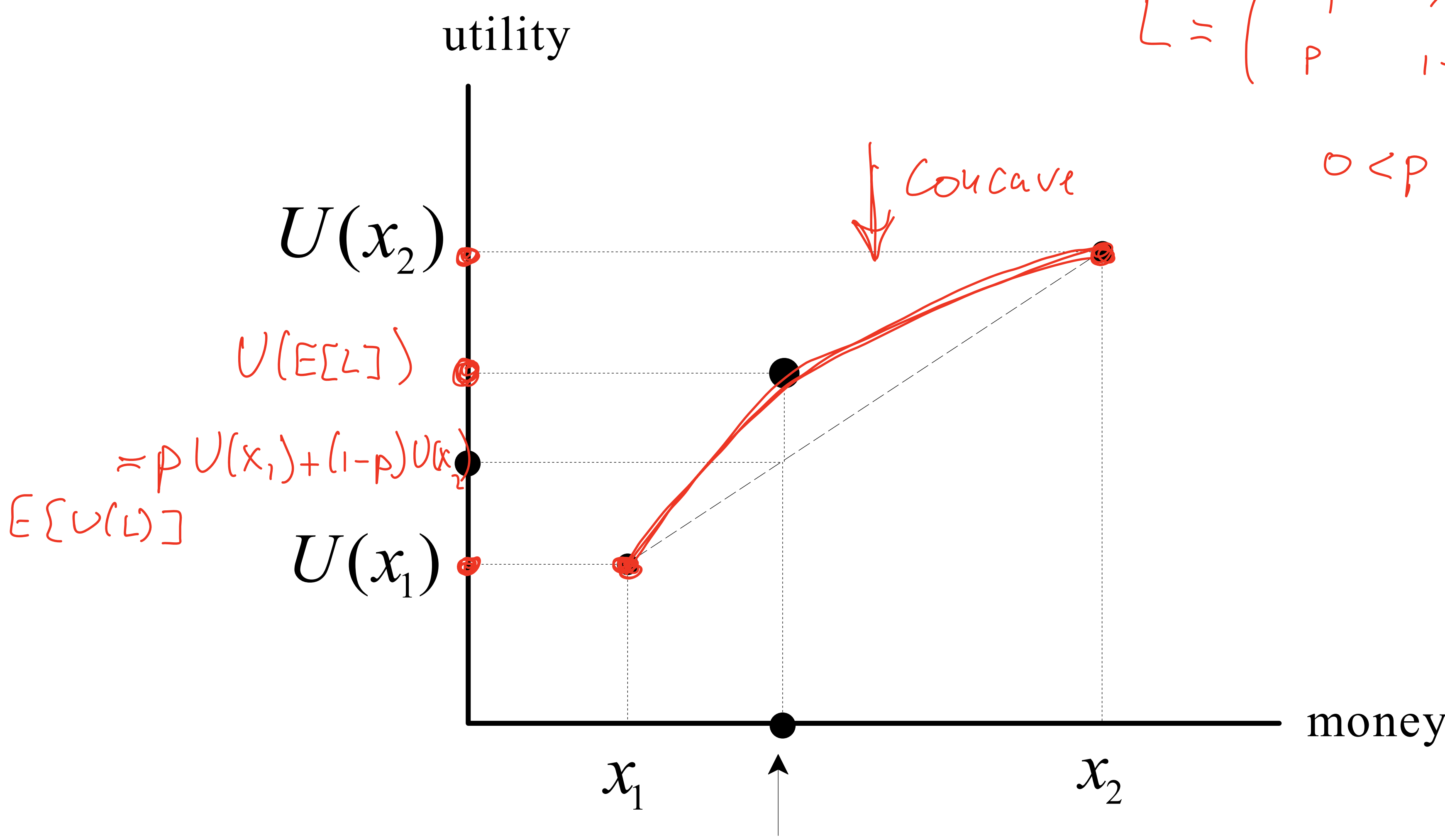
- $U(\$x) = x^2$. Then RISK LOVING $E[U(L)] = \frac{1}{4}16^2 + \frac{1}{2}36^2 + \frac{1}{4}64^2$

$U(38) = 38^2 = 1{,}444$ $<$ $= 1{,}736$

Risk averse: $U(E[L]) > E[U(L)]$

$$L = \begin{pmatrix} x_1 & x_2 \\ p & 1-p \end{pmatrix}$$

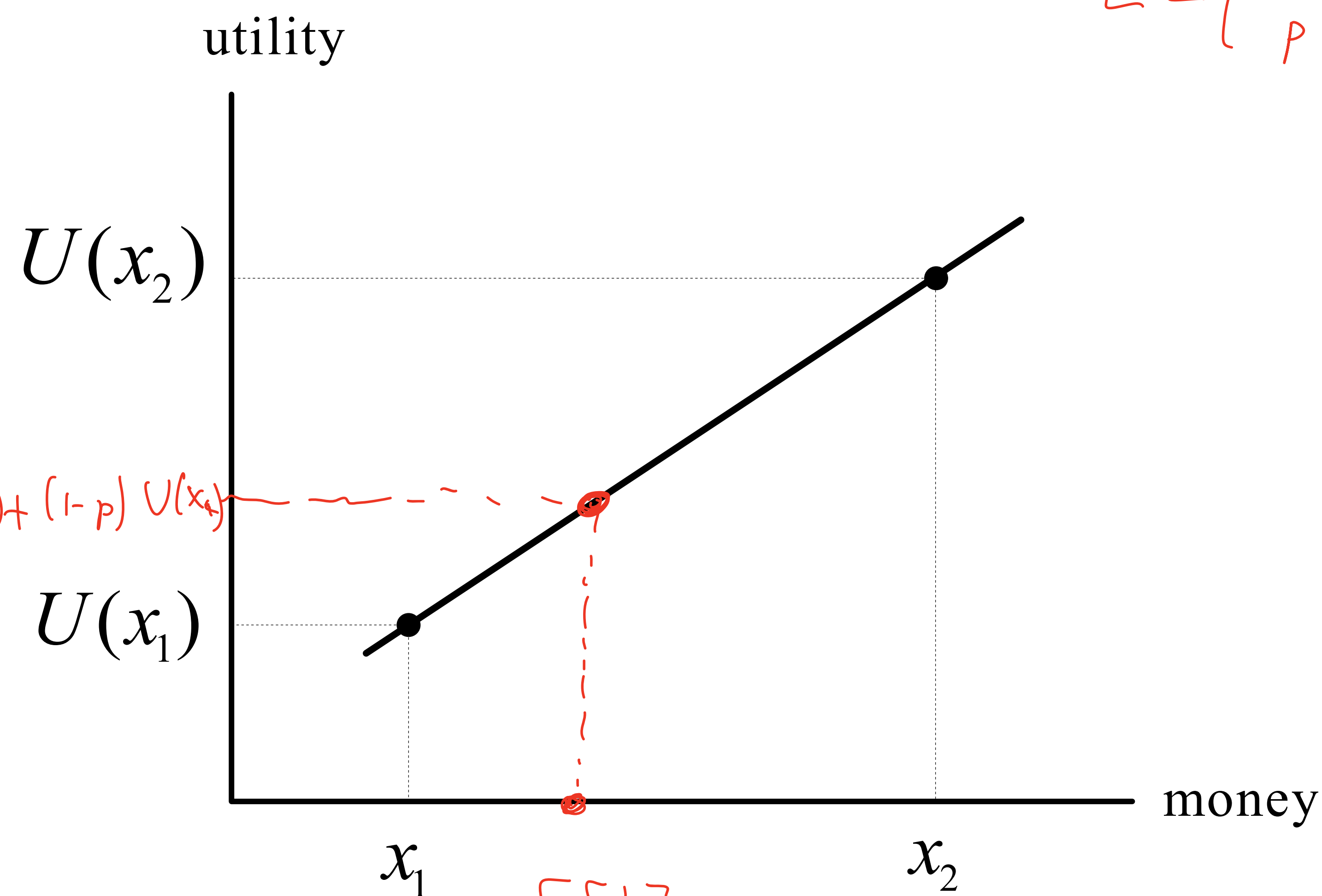
$0 < p < 1$



$px_1 + (1-p)x_2 = E[L]$

Risk neutrality: straight line

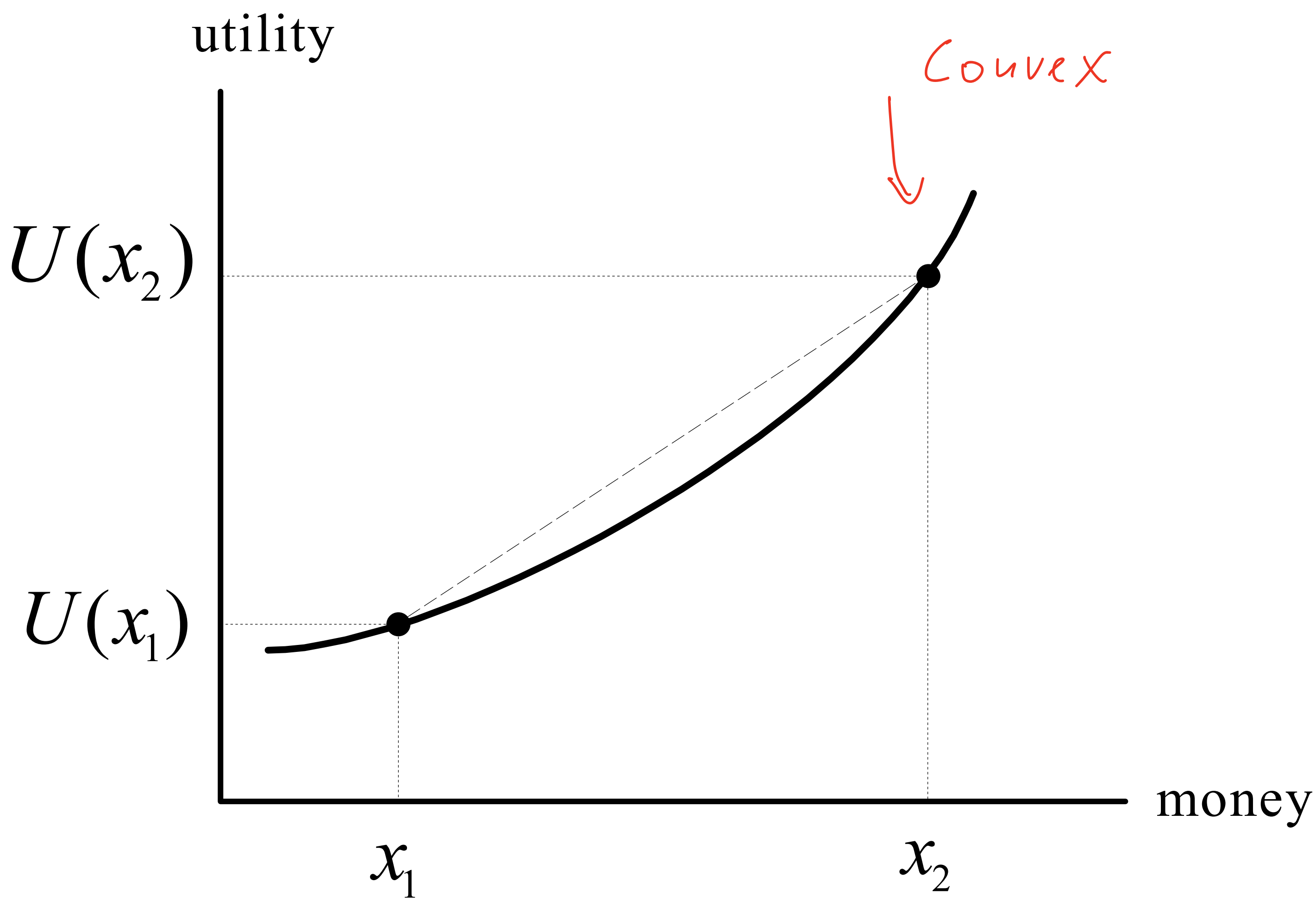
$$L = \begin{pmatrix} x_1 & x_2 \\ p & 1-p \end{pmatrix}$$



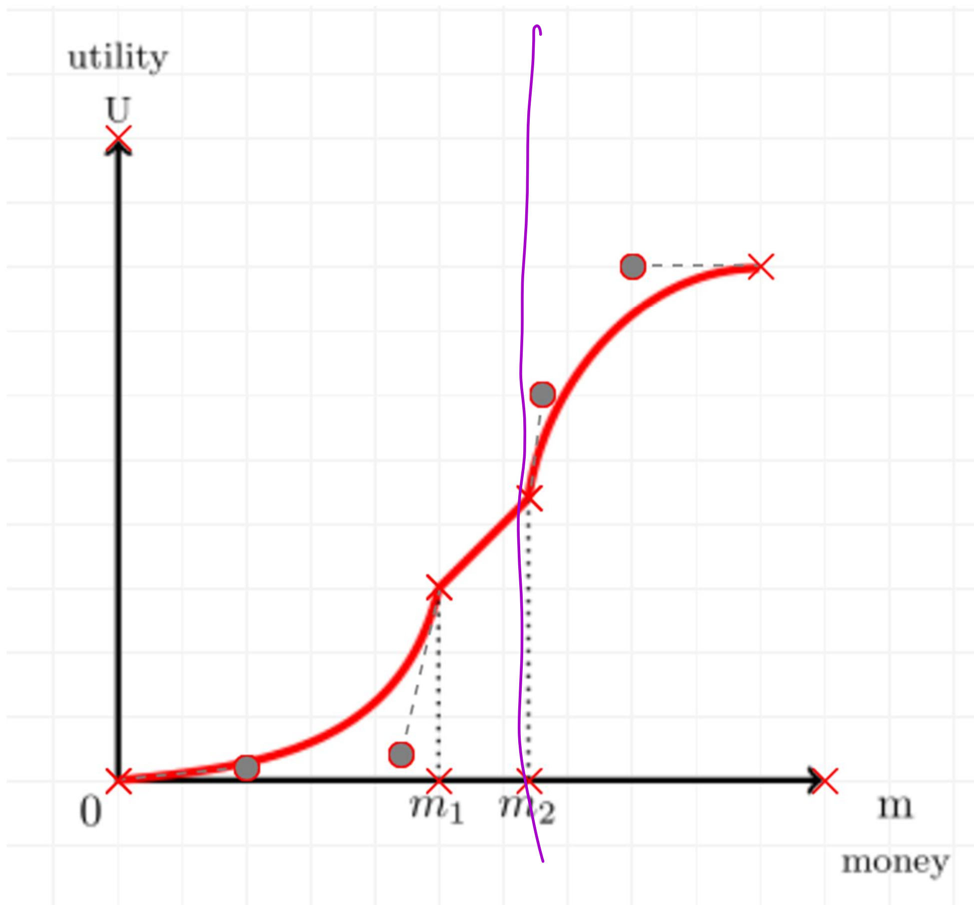
$E[U(L)] = pU(x_1) + (1-p)U(x_2)$

$E[L] = px_1 + (1-p)x_2$

Risk loving: strictly convex



Mixture of attitudes:



A function $U(x)$ is strictly concave if and only if $\frac{d^2U}{dx^2} < 0$

Examples:

$$U(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$U'(x) = \frac{1}{2} x^{-\frac{1}{2}}$$

$$U''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-\frac{3}{2}}$$

$$= -\frac{1}{4} \frac{1}{x^{\frac{3}{2}}} = -\frac{1}{4\sqrt{x^3}} < 0$$

$$U(x) = \ln(x)$$

$$U'(x) = \frac{1}{x} = x^{-1}$$

$$U''(x) = (-1)\, x^{-2} =$$

$$= -\frac{1}{x^2} < 0$$

Both incorporate risk aversion

A function $U(x)$ is strictly convex if and only if $\frac{d^2U}{dx^2} > 0$

Examples:

$$U(x) = x^2 \qquad\qquad U(x) = e^x$$

$$U'(x) = 2x \qquad\qquad U'(x) = e^x$$

$$U''(x) = 2 > 0 \qquad\qquad U''(x) = e^x > 0$$

A function $U(x)$ is has a straight-line graph if and only if $\dfrac{dU^2}{dx^2} = 0$

$$U(x) = a + bx$$

$$U'(x) = b$$

$$U''(x) = 0$$