Published in: *Mathematical Social Sciences*, 36, 1998, pp. 3-29.

ASSESSING THE TRUTH AXIOM UNDER INCOMPLETE INFORMATION*

Giacomo Bonanno

and

Klaus Nehring

Department of Economics,
University of California,
Davis, CA 95616-8578

Abstract

Within an incomplete information framework (where the primitives are the individuals' belief hierarchies) the Truth Axiom is stated locally as the hypothesis that no individual has any false beliefs and that this fact is common belief. We decompose this "Truth Condition" into three parts: Truth *of* common belief, Truth *about* common belief and "quasi-coherence" (the common possibility of common belief in no error). We show the latter to be equivalent to the absence of unbounded gains from betting for "moderately risk averse" agents as well as to a qualitative impossibility of "agreeing to disagree". We show that these characterizations may not hold in more general non-Bayesian models of decision-making under uncertainty.

*We are grateful to an anonymous referee for detailed and helpful comments.

Corresponding author: Giacomo Bonanno, Department of Economics, University of California, One Shields Avenue, Davis, CA 95616-8578, USA.
E-mail: gfbonanno@ucdavis.edu, Fax: 530 – 752 9382, Tel. 530 – 752 1574.

1. Introduction

The structures that are most often used in the economics and computer science literature to discuss interactive beliefs / knowledge are partition structures [1]. Partition structures embody the S5 logic for individual beliefs, in particular the Truth Axiom, that is, the assumption that it is a *necessary* truth (true in all possible worlds of the model) that no one has any false beliefs. As Stalnaker (1994, 1996) points out there is an important conceptual difference between a theory that builds S5 into the concept of knowledge (which − Stalnaker argues − is based on equivocating between knowledge and belief) and a theory that describes epistemic conditions under which knowledge and belief coincide, and then considers the consequences of assuming those conditions. In the latter the Truth Axiom can be expressed *locally* (that is, as a property of the individuals' belief hierarchies) as the condition that no one has any false beliefs and that it is common belief that no one has any false beliefs. Let **T** represent the event that no one has false beliefs and $B_*\mathbf{T}$ the event that it is common belief that no one has false beliefs (B_* denotes the common belief operator). Then the Truth Axiom holds locally for a given profile of belief hierarchies if and only if $\tau \in \mathbf{T} \cap B_*\mathbf{T}$, where τ is the true state of the model that represents those belief hierarchies; we will say that at τ the *Truth Condition* is satisfied in this case.

In applications to game theory and economics (see, for example, Ben Porath, 1997, Morris, 1994, Stalnaker, 1994, 1996, Stuart, 1997) the crucial assumption is $B_*\mathbf{T}$, that is, common belief in no error. How should this assumption be assessed? One could take the position that correctness of probability-one beliefs is empirically highly plausible and that therefore common belief in no error is a reasonable hypothesis. In this paper we propose an alternative view. Let $\mathbf{T}_i$ be the event that individual i has correct beliefs and $B_i\mathbf{T}_i$ the event that individual i believes that she has correct beliefs (B_i denotes the belief operator of individual i); $B_i\mathbf{T}_i$ will be referred to as the property of *secondary reflexivity* of individual beliefs. For a single agent the assumption of secondary reflexivity is not an empirical assumption but rather a logical property of beliefs: an individual cannot coherently assign positive probability to the event that she assigns probability one to something which is false. The assumption of common belief in no error ($B_*\mathbf{T}$) can be viewed as an intersubjective generalization of the logical requirement of secondary reflexivity of individual beliefs

($B_i\mathbf{T}_i$). To explore this interpretation we consider two related types of intersubjective conditions: *qualitative* agreement and the absence of *unbounded* gains from betting.

In Section 3 we study Aumann's (1976) notion of "agreeing to disagree" from the qualitative point of view of this paper, which is concerned with the intersubjective structure of individuals' *certain* (probability one) beliefs. We show (Proposition 3) that the absence of agreeing to disagree about "union consistent" qualitative belief indices is equivalent to, not $B_*\mathbf{T}$, but the common possibility of it, that is, $\neg B_*\neg B_* T$; we call this property *quasi-coherence* of beliefs[2]. Thus the notion of Agreement provides a partial clarification of the intersubjective implications of common belief in no error.

In Section 4 we take a (Bayesian) *behavioral* point of view and consider the betting implications of the property of common belief in no error. The suspicion is that if some individual attributes false probability-one beliefs to another individual then extreme forms of betting might arise. In the case of complete information (defined by the condition that the beliefs of each individual are common belief) one can make this intuition precise by considering betting in the context of "moderately risk-averse" preferences. Indeed, in this context, common belief in no error fails at a state if and only if *unbounded* gains from betting are possible. This fact generalizes only in a limited way to situations of *incomplete* information. In Proposition 4 we show that at state α unbounded gains from betting are impossible if and only if $\alpha \in \neg B_*\neg B_* T$. Thus the relevant property in general is, again, quasi-coherence of beliefs rather than common belief in no error.

In Section 5 the Truth Condition is decomposed into three heterogeneous properties: Truth *of* common belief (what is commonly believed is true), common belief in Truth *about* common belief (if an individual believes that E is common belief, then E is indeed commonly believed) and quasi-coherence of beliefs. Only the latter has any agreement-type implications. Since from an economic point of view the absence of unbounded gains from betting seems a very compelling assumption, and by themselves the other two assumptions are significantly weaker than the Truth Condition, this proposition lends support to the latter.

In the concluding section we point out that this justification for common belief in no error hinges on a Bayesian definition of certain beliefs. Within more general models of decision making under uncertainty, absence of unbounded gains from trade no longer implies quasi-coherence since an individual's failure to be certain of some event E no longer necessarily entails a willingness to bet *against E* even at extremely favorable odds. We illustrate this in an example in which agents' preferences can be represented by both the multiple-prior and the Choquet expected utility models.

All proofs are given in the appendix.

2. Interactive belief frames

DEFINITION 1. A *KD45 frame for interactive beliefs* (or frame, for short) is a tuple

$$\mathcal{F} = \langle \mathrm{N},\ \Omega,\ \tau,\ \{\mathrm{I}_i\}_{i\in \mathrm{N}} \rangle$$

where

- $\mathrm{N} = \{1, ..., n\}$ is a finite set of *individuals*.
- Ω is a finite set of *states* (or possible worlds). The subsets of Ω are called *events.*
- $\tau \in \Omega$ is the "true" or "actual" state[3].
- for every individual $i \in \mathrm{N}$, $\mathrm{I}_i : \Omega \to 2^{\Omega} \backslash \varnothing$ (where 2^{Ω} denotes the set of subsets of Ω) is i's *possibility correspondence* satisfying the following properties (whose interpretation is given in Remark 3): $\forall\, \alpha, \beta \in \Omega$,

  Transitivity: if $\beta \in \mathrm{I}_i(\alpha)$ then $\mathrm{I}_i(\beta) \subseteq \mathrm{I}_i(\alpha)$,

  Euclideanness: if $\beta \in \mathrm{I}_i(\alpha)$ then $\mathrm{I}_i(\alpha) \subseteq \mathrm{I}_i(\beta)$.

For every $\alpha \in \Omega$, $\mathrm{I}_i(\alpha)$ represents the set of states that individual i considers possible at α.

REMARK 1. The assumption of finiteness of Ω, common in the economics and computer science literature[4], is made for technical convenience. Of the results given below, Propositions 3 and 5 apply in fact to the general case where Ω may be infinite.

REMARK 2 (Graphical representation). A non-empty-valued and transitive possibility correspondence $\mathrm{I} : \Omega \to 2^{\Omega} \backslash \varnothing$ can be uniquely represented (see Figures 1-4) as an asymmetric directed graph[5] whose vertex set consists of disjoint events (called *cells* and represented as rounded rectangles) and states, and each arrow goes from, or points to, either a cell or a state that does not

belong to a cell. In such a directed graph, $\omega' \in I(\omega)$ if and only if either ω and ω' belong to the same cell or there is an arrow from ω, or the cell containing ω, to ω', or the cell containing ω'.
Conversely, given a transitive directed graph in the above class such that each state either belongs to a cell or has an arrow out of it, there exists a unique non-empty-valued, transitive possibility correspondence which is represented by the directed graph.
The possibility correspondence is euclidean if and only if all arrows connect states to cells and no state is connected by an arrow to more than one cell (for an example of a non-Euclidean possibility correspondence see the common possibility correspondence I_* of Figure 1 below).
Finally, if − in addition − the possibility correspondence is reflexive ($\omega \in I(\omega)$, $\forall\, \omega \in \Omega$), then one obtains a partition model where each state is contained in a cell and there are no arrows between cells.

Given a frame and an individual i, i's *belief operator* $B_i : 2^\Omega \to 2^\Omega$ is defined as follows: $\forall\, E \subseteq \Omega$, $B_i E = \{\omega \in \Omega : I_i(\omega) \subseteq E\}$. $B_i E$ can be interpreted as the event that (i.e. the set of states at which) individual i *believes* that event E has occurred.

REMARK 3. A belief operator $B : 2^\Omega \to 2^\Omega$ is *normal* if satisfies the following properties: $\forall\, E,F \subseteq \Omega$,

Necessity: $B\Omega = \Omega$

Conjunction: $B(E \cap F) = BE \cap BF$

Monotonicity: if $E \subseteq F$ then $BE \subseteq BF$.

It is well-known that the belief operator derived from a possibility correspondence is normal. Instead of taking possibility correspondences as primitives, one could start with a normal belief operator B_i for every individual i and derive from it i's possibility correspondence
$I_i : \Omega \to 2^\Omega$ as follows: $I_i(\alpha) = \{\omega \in \Omega : \alpha \in \neg B_i \neg\{\omega\}\}$. These two maps are one the inverse of the other. Because of this equivalence, properties of the belief operators can be characterized in terms of properties of the possibility correspondences. For example, it is well known (see Chellas, 1984, p. 164) that non-empty-valuedness of I_i is equivalent to *consistency* of beliefs: $\forall\, E \subseteq \Omega$, $B_i E \subseteq \neg B_i \neg E$ (an individual cannot simultaneously believe E and not E; for every event F, $\neg F$ denotes the

complement of F); transitivity of I_i is equivalent to *positive introspection* of beliefs: $\forall E \subseteq \Omega,\ B_iE \subseteq B_iB_iE$ (if the individual believes E then she believes that she believes E); finally, euclideanness of I_i is equivalent to *negative introspection* of beliefs: $\forall E \subseteq \Omega,\ \neg B_iE \subseteq B_i\neg B_iE$ (if the individual does not believe E, then she believes that she does not believe E).

Notice that we have allowed for false beliefs by not assuming reflexivity of the possibility correspondences, which – as is well known (Chellas, 1984, p. 164) – is equivalent to the *Truth Axiom*: $\forall E \subseteq \Omega,\ B_iE \subseteq E$ (if the individual believes E then E is indeed true).

The *common belief operator* B_* is defined as follows. First, for every $E \subseteq \Omega$, let $B_eE = \bigcap_{i \in N} B_iE$, that is, B_eE is the event that everybody believes E. The event that E is commonly believed is defined as the infinite intersection:

$$B_*E = B_eE \cap B_eB_eE \cap B_eB_eB_eE \cap \ldots$$

The corresponding *common possibility correspondence* $I_*: \Omega \to 2^{\Omega}\setminus\varnothing$ is given by: for every $\alpha \in \Omega$, $I_*(\alpha) = \{\omega \in \Omega : \alpha \in \neg B_*\neg\{\omega\}\}$. It is well known[6] that I_* can be characterized as the *transitive closure* of $\bigcup_{i \in N} I_i$, that is,

> $\forall \alpha, \beta \in \Omega,\ \ \beta \in I_*(\alpha)$ if and only if there is a sequence $\langle i_1, \ldots i_m \rangle$ in N and a sequence $\langle \eta_0, \eta_1, \ldots, \eta_m \rangle$ in Ω such that: (i) $\eta_0 = \alpha$, (ii) $\eta_m = \beta$ and (iii) for every $k = 0, \ldots, m-1$, $\eta_{k+1} \in I_{i_{k+1}}(\eta_k)$.

Note that, although I_* is always non-empty-valued and transitive, in general it need not be euclidean (despite the fact that the individual possibility correspondences are: for an example see Figure 1; recall that – cf. Remark 3 – I_* is euclidean if and only if B_* satisfies Negative Introspection: $\forall E \subseteq \Omega,\ \neg B_*E \subseteq B_*\neg B_*E$).

Given a frame $\mathcal{F}$, a *model based on* $\mathcal{F}$ is a tuple $\langle \mathcal{F}, \Psi, \phi \rangle$, where

- Ψ is a set of *external circumstances* or *facts of nature*, and
- $\phi : \Omega \to 2^{\Psi}$ (where 2^{Ψ} is the set of subsets of Ψ) is a function that specifies, for every state, the facts of nature that are true at that state.

A state in a model determines, for each individual, her beliefs about the external world (her first-order beliefs), her beliefs about the other individuals' beliefs about the external world (her second-order beliefs), her beliefs about their beliefs about her beliefs (her third-order beliefs), and so on, *ad infinitum*. An entire hierarchy of beliefs about beliefs about beliefs ... about the relevant facts is thus encoded in each state of an interactive belief model. For example, consider the following model, which is illustrated in Figure 1 according to the convention established in Remark 2: N = {1, 2}, $\Omega = \{\tau, \beta, \gamma\}$, Ψ = {sunny, cloudy}, $\phi(\tau) = \phi(\gamma)$ = {sunny}, $\phi(\beta)$ = {cloudy}, $I_1(\tau) = \{\tau\}$, $I_1(\beta) = I_1(\gamma) = \{\beta\}$, $I_2(\tau) = I_2(\gamma) = \{\tau, \gamma\}$, $I_2(\beta) = \{\beta\}$. Here the true state τ describes a world where in fact it is sunny and both individuals correctly believe that it is sunny; however, while individual 1 believes that individual 2 believes that it is sunny, individual 2 is uncertain as to whether 1 believes that it is sunny or he believes that (it is common belief that) it is cloudy, etc.

Insert Figure 1

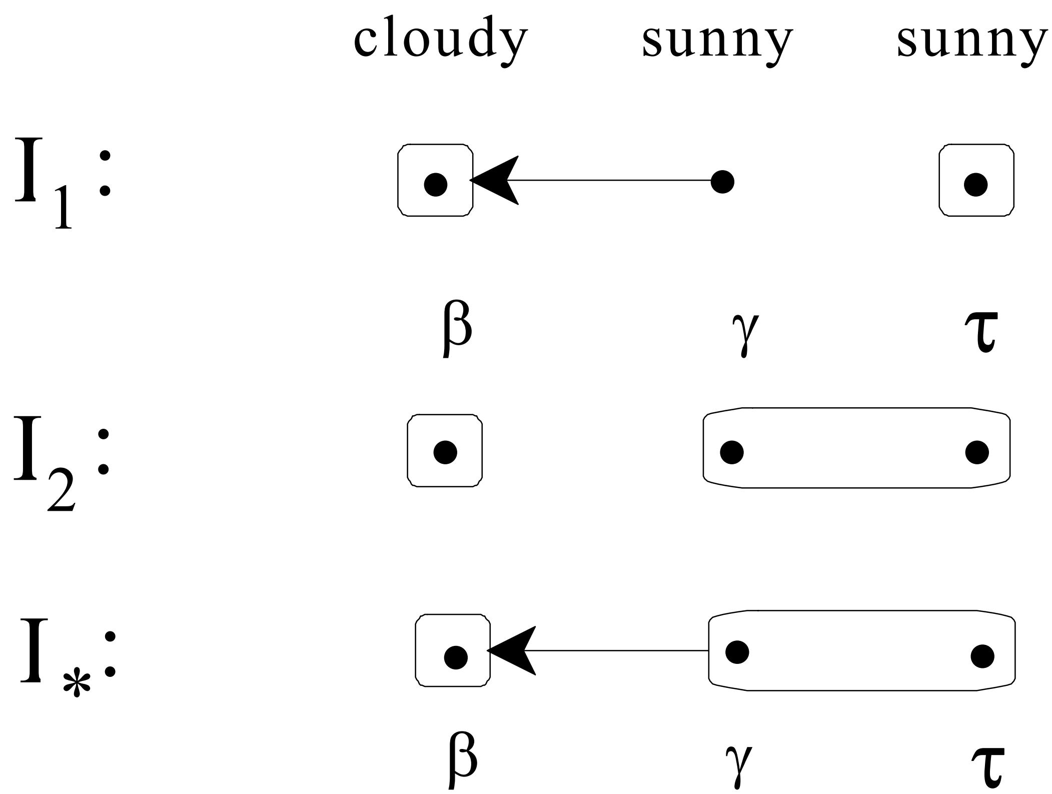

Figure 1

N = {1, 2}, Ω = {τ, β, γ}, $I_1(\tau) = \{\tau\}$, $I_1(\beta) = I_1(\gamma) = \{\beta\}$, $I_2(\tau) = I_2(\gamma) = \{\tau, \gamma\}$, $I_2(\beta) = \{\beta\}$, $I_*(\tau) = I_*(\gamma) = \{\beta, \gamma, \tau\}$, $I_*(\beta) = \{\beta\}$, Ψ = {sunny, cloudy}, $\phi(\tau) = \phi(\gamma)$ = {sunny}, $\phi(\beta)$ = {cloudy}.

Let $\mathbf{T_j}$ (for Truth of j's beliefs) be the following event:

$$\mathbf{T_j} = \bigcap_{E \in 2^{\Omega}} \neg(B_j E \cap \neg E)$$

Thus, for every $\alpha \in \Omega$, $\alpha \in \mathbf{T_j}$ if and only if individual j does not have any false beliefs at α (for every $E \subseteq \Omega$, if $\alpha \in B_j E$ then $\alpha \in E$)[7]. Let **T** (for Truth) be the event that no individual has any false beliefs:

$$\mathbf{T} = \bigcap_{j \in N} \mathbf{T_j} .$$

For example, in the frame of Figure 1, $\mathbf{T} = \{\beta, \tau\}$ and, therefore, $B_* \mathbf{T} = \{\beta\}$.

DEFINITION 2. For every $\alpha \in \Omega$, the *Truth Condition holds at* α if and only if

$$\alpha \in \mathbf{T} \cap B_*\mathbf{T}.$$

The above definition is justified by the following observation. Given a frame $\langle N, \Omega, \tau, \{I_i\}_{i\in N} \rangle$, define the *τ-reduced frame* as the frame $\langle N, \Omega', \tau, \{I'_i\}_{i\in N} \rangle$ where $\Omega' = I_*(\tau) \cup \{\tau\}$ and I'_i is the restriction of I_i to Ω'. Let B'_i be the corresponding belief operator of individual i and I'_* the corresponding common possibility correspondence. Then I'_* is the restriction of I_* to Ω' [in particular, $I'_*(\tau) = I_*(\tau)$] and for every $E' \subseteq \Omega'$ $B'_i E' = B_i E' \cap \Omega'$. If $\langle N, \Omega, \tau, \{I_i\}_{i\in N} \rangle$ is a frame where $\tau \in T \cap B_*\mathbf{T}$, then in the τ-reduced frame the following is true: $\forall i \in N, \forall E' \subseteq \Omega', B'_i E' \subseteq E'$ (note, however, that in the original frame in general it is not true that $\forall i \in N, \forall E \subseteq \Omega, B_i E \subseteq E$: see Figure 2a). Thus the τ-reduced frame is a partitional frame (unlike the original frame, in general). Figure 2b shows the τ-reduced frame corresponding to the frame of Figure 2a.

Insert Figure 2

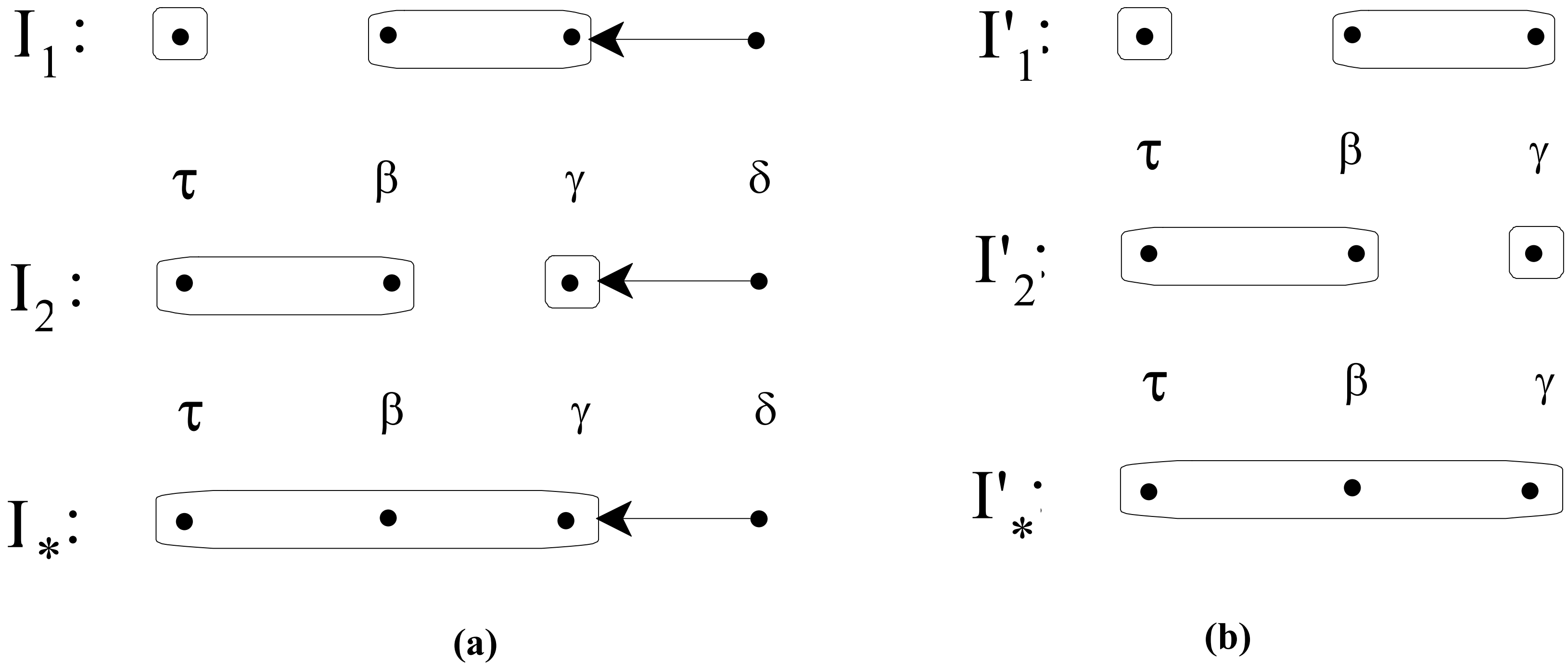

Figure 2

A frame (a) and its τ-reduced form (b).

The intersubjective implications of the Truth Axiom ($\forall$ i$\in$N, $\forall$ E$\subseteq \Omega$, $B_iE \subseteq$ E) are strong:

> "The assumption that Alice believes (with probability one) that Bert believes (with probability one) that the cat ate the canary tells us nothing about what Alice believes about the cat and the canary themselves. But if we assume instead that Alice *knows* that Bert *knows* that the cat ate the canary, it follows, not only that the cat in fact ate the canary, but that Alice knows it, and therefore believes it as well" (Stalnaker, 1996, p. 153).

This observation can be stated as a local property of beliefs.

DEFINITION 3. Given two individuals, i and j, and a state α, *i is like-minded with j at α* if and only if i shares all the beliefs that she attributes to j, that is, for every event E, if $\alpha \in B_iB_jE$ then $\alpha \in B_iE$. Let $\mathbf{L}_{ij}$ be the event that i is like minded with j:

$$\mathbf{L}_{ij} = \bigcap_{E\in 2^{\Omega}} \neg\,(B_iB_jE \cap \neg B_iE).$$

Let **L** be the event that every individual is like-minded with every other individual:

$$\mathbf{L} = \bigcap_{i\in N} \bigcap_{j\in N} \mathbf{L}_{ij}\,.$$

The following equivalence exhibits a converse to Stalnaker's observation. It is a straightforward consequence of secondary reflexivity.

OBSERVATION 1. $\mathbf{L}_{ij} = B_i\mathbf{T}_j$ and, therefore, $B_*\mathbf{L} = B_*\mathbf{T}$.

DEFINITION 4. Define *quasi-coherence* of beliefs as the common *possibility* of public like-mindedness and denote it by **Q**:

$$\mathbf{Q} = \neg B_*\neg B_*\mathbf{L} = \neg B_*\neg B_*\mathbf{T}.$$

In the following section we provide a characterization of quasi-coherence in terms of "agreeing to disagree" about qualitative belief indices. In Section 4 we show that quasi-coherence is also equivalent to the impossibility of unbounded gains from betting in environments where the individuals are "moderately risk averse".

3. Agreeing to disagree and quasi-coherence

The notion of "agreeing to disagree" was introduced by Aumann (1976) in a Bayesian context. Aumann assumed the S5 logic for individual beliefs (information partitions) and showed that if the beliefs of the individuals are obtained from a common prior probability distribution over Ω by updating on private information, then the individuals cannot agree to disagree in the following sense: if it is common knowledge that individual 1's posterior of some event E is p and 2's posterior of E is q, then p = q[8]. In this section we extend the notion of Agreement by considering a class of qualitative belief indices within a Bayesian context and show that the absence of agreeing to disagree with respect to those indices is equivalent to quasi-coherence.

For purposes of notational simplification, throughout this section we restrict attention to the case where N = {1,2}.

DEFINITION 5. A *Bayesian frame based on the frame* $\mathcal{F}$ is a tuple $\mathcal{B} = \langle \mathcal{F}, \{p_i\}_{i\in N} \rangle$ where

- for every individual $i \in N$, $p_i : \Omega \to \Delta(\Omega)$ ($\Delta(\Omega)$ denotes the set of probability distributions over Ω) is a function that specifies i's *probabilistic beliefs,* satisfying the following properties[9] [we use the notation $p_{i,\alpha}$ rather than $p_i(\alpha)$]: $\forall\ \alpha,\beta \in \Omega$,

  (i) $\text{supp}(p_{i,\alpha}) = I_i(\alpha)$, and

  (ii) if $I_i(\alpha) = I_i(\beta)$ then $p_{i,\alpha} = p_{i,\beta}$.

Thus $p_{i,\alpha} \in \Delta(\Omega)$ is individual i's subjective probability distribution at state α and the above two conditions say that every individual knows her own beliefs. Let $\|p_i = p_{i,\alpha}\|$ denote the event $\{\omega \in \Omega : p_{i,\omega} = p_{i,\alpha}\}$ and $\|I_i = I_i(\alpha)\|$ the event $\{\omega \in \Omega : I_i(\omega) = I_i(\alpha)\}$. It is clear that

$\|p_i = p_{i,\alpha}\| = \|I_i = I_i(\alpha)\|$ and that the set $\{\|p_i = p_{i,\omega}\| : \omega \in \Omega\}$ is a partition of Ω; it is often referred to as individual i's *type partition*.

Given a Bayesian frame, a *belief index* is a function $f : \Delta(\Omega) \to X$ (where X is a set with at least two elements). A *proper* belief index is one that satisfies the following property: $\forall\, p, q \in \Delta(\Omega),\ \forall x \in X, \forall a \in [0,1]$, if $f(p) = f(q) = x$ then $f(ap + (1-a)q) = x$.[10] Properness is necessary in order to ensure that commonly believed inequality of indices can be interpreted as disagreement (see Bonanno and Nehring, 1996). Let $\mathcal{F}_\infty$ denote the class of proper belief indices.

Given a proper belief index $f : \Delta(\Omega) \to X$ and an individual $i \in N$, define $f_i : \Omega \to X$ by $f_i(\alpha) = f(p_{i,\alpha})$. For every $x \in X$ denote the event $\{\alpha \in \Omega : f_i(\alpha) = x\}$ by $\|f_i = x\|$.

DEFINITION 6. Given a Bayesian frame and a proper belief index $f: \Delta(\Omega) \to X$, at $\alpha \in \Omega$ there is *Agreement for f* or *f-Agreement* if and only if, for all $x_1, x_2 \in X$,

$$\text{if} \quad \alpha \in B_*\left(\|f_1 = x_1\| \cap \|f_2 = x_2\|\right) \quad \text{then} \quad x_1 = x_2, \tag{1}$$

that is, if at α it is common belief that individual 1's belief index is x_1 and individual 2's index is x_2, then the two indices must be equal. Let ***f*-Agree** be the event that there is f-Agreement:

$$\boldsymbol{f}\textbf{-Agree} = \bigcap_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \neg B_*\left(\|f_1 = x_1\| \cap \|f_2 = x_2\|\right). \tag{2}$$

Given a set $\mathcal{F}$ of proper belief indices, at $\alpha \in \Omega$ there is *Agreement on* $\mathcal{F}$ or $\mathcal{F}$*-Agreement* if and only if, $\forall f \in \mathcal{F}$, $\alpha \in$ ***f*-Agree**. Let $\mathcal{F}$**-Agree** be the event that there is $\mathcal{F}$-Agreement:

$$\mathcal{F}\textbf{-Agree} = \bigcap_{f \in \mathcal{F}} \boldsymbol{f}\textbf{-Agree}. \tag{3}$$

Of particular interest are the following special cases of proper belief indices: *simple* indices, which take on only two values, 0 and 1, and *qualitative* indices, which depend only on the support of $p \in \Delta(\Omega)$. We denote the first class by $\mathcal{F}_2$ and the latter by $\mathcal{F}_Q$. Thus

$$\mathcal{F}_2 = \{ f: \Delta(\Omega) \to X : \text{ (i) } f \in \mathcal{F}_\infty, \text{ (ii) } X = \{0, 1\} \text{ and (iii) } f^{-1}(1) \text{ is closed}^{11} \} \tag{4}$$

$$\mathcal{F}_Q = \{ f \in \mathcal{F}_\infty : \forall\, p, q \in \Delta(\Omega), \text{ if } \mathrm{supp}(p) = \mathrm{supp}(q) \text{ then } f(p) = f(q) \}. \tag{5}$$

The following proposition, which is proved in Bonanno and Nehring (1996), follows from a straightforward separation argument.

PROPOSITION 1. $f \in \mathcal{F}_2$ if and only if there exists a random variable $Y : \Omega \to \mathbb{R}$ such that, for all $p \in \Delta(\Omega)$, $f(p) = \begin{cases} 1 & \text{if } \sum_{\omega \in \Omega} Y(\omega)\, p(\omega) \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

REMARK 4. A qualitative belief index can be written as $f = d_f \circ \mathrm{supp}$, with $d_f : 2^\Omega \setminus \varnothing \to X$ (such functions d_f have been studied in Rubinstein and Wolinsky, 1990). A qualitative belief index is proper if and only if d_f is *union consistent*, that is,

$$\forall\, m \geq 1,\ \forall E_1, \ldots, E_m \in 2^\Omega,\ \forall x \in X, \quad \text{if } d_f(E_k) = x \text{ for all } k = 1, \ldots, m \text{ then } d_f\Big(\bigcup_{k=1}^m E_k\Big) = x. \tag{6}$$

Note that since the events $E_1, \ldots, E_m$ are *not* assumed to be pairwise disjoint, union consistency is a stronger property than the Sure Thing Principle defined by Bacharach (1985).

Fix an event $E \neq \varnothing$ and consider the following index: $f_E(p) = \begin{cases} 1 & \text{if } \mathrm{supp}(p) \subseteq E \\ 0 & \text{otherwise} \end{cases}$.

Thus, for every individual i and state α, $f_E(p_{i,\alpha}) = 1$ if and only if $\alpha \in B_i E$[12]. Let $\mathcal{F}_S$ = $\{f_E : \Delta(\Omega) \to \{0,1\} : E \subseteq \Omega\}$. Clearly, $\mathcal{F}_S \subseteq \mathcal{F}_2 \cap \mathcal{F}_Q$. The following proposition shows that in fact $\mathcal{F}_S$ coincides with $\mathcal{F}_Q \cap \mathcal{F}_2$.

PROPOSITION 2. $\mathcal{F}_S = \mathcal{F}_Q \cap \mathcal{F}_2$.

Note that $\alpha \in \mathcal{F}_S$-**Agree** if and only if, for no event E, $\alpha \in B_*(B_1E \cap \neg B_2E)$, that is, there is no event about which the two individuals "agree to disagree":

$$\mathcal{F}_S\text{-}\mathbf{Agree} = \bigcap_{i \in N} \bigcap_{j \in N} \bigcap_{E \in 2^\Omega} \neg B_*(B_iE \cap \neg B_jE).$$

LEMMA 1. $\forall \alpha \in \Omega, \; \alpha \in \mathcal{F}_S$-**Agree** if and only if

$$\forall i,j \in N, \; \exists \beta \in I_*(\alpha) \text{ such that } I_j(\beta) \subseteq \bigcup_{\omega \in I_*(\alpha)} I_i(\omega).$$ [13]

As a corollary to Lemma 1 we get that quasi-coherence of beliefs rules out agreeing to disagree about events.

COROLLARY 1. $\mathbf{Q} \subseteq \mathcal{F}_S$-**Agree**.

The converse to Corollary 1 does not hold. To see this, consider the frame illustrated in Figure 3. At the true state τ both individuals correctly believe p; however, while individual 2 believes that individual 1 believes p, individual 1 considers it possible that 2 is in doubt as to whether or not p is true. By Lemma 1, $\mathcal{F}_S$-**Agree =** Ω; on the other hand, $\mathbf{Q} = \varnothing$ (in fact, $\mathbf{T} = \{\tau, \delta\}$ and, therefore, $B_*\mathbf{T} = \varnothing$; thus $\neg B_*\neg B_*\mathbf{T} = \varnothing$).

Insert Figure 3

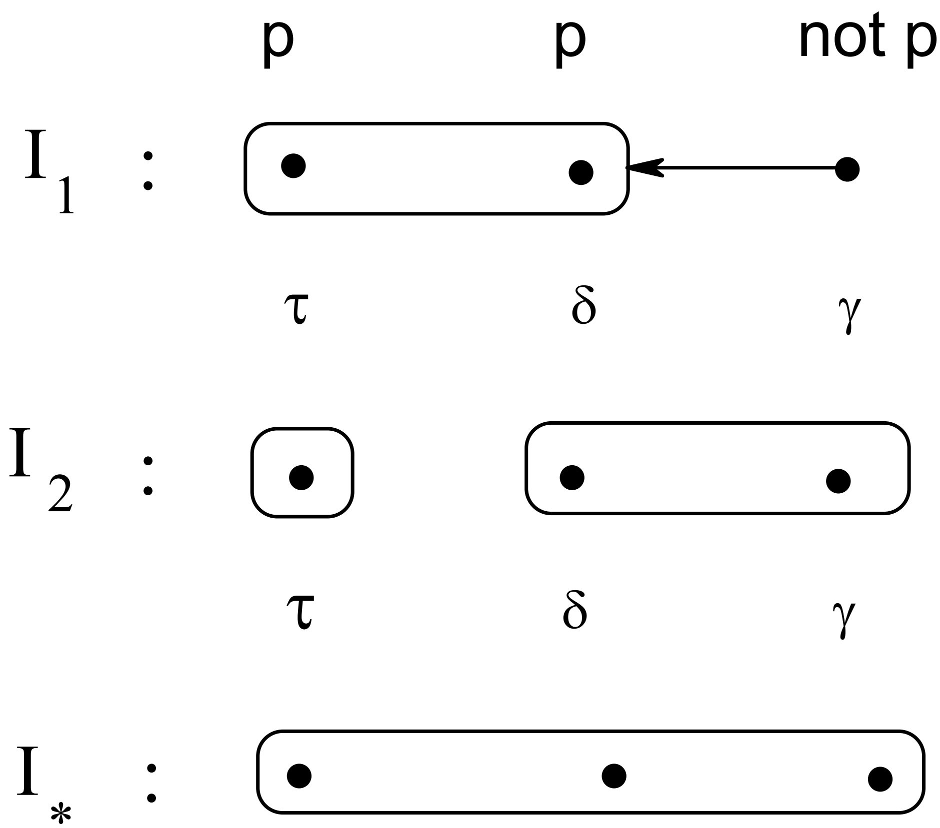

Figure 3

A frame that shows that the converse of Corollary 1 does not hold

To obtain a full characterization of quasi-coherence one needs to consider the *entire* class of qualitative belief indices.

PROPOSITION 3. **Q** $= \mathcal{F}_Q$**-Agree**.

It follows from Proposition 3 and the above example that $\mathcal{F}_S$**-Agree** $\neq$ $\mathcal{F}_Q$**-Agree**. Thus, in contrast to the case of general "quantitative" proper belief indices, for which simplicity can be assumed without loss of generality (i.e. $\mathcal{F}_\infty$**-Agree** = $\mathcal{F}_2$**-Agree**: see Bonanno and Nehring, 1996), simplicity *is* a restrictive assumption for qualitative belief indices.

4. Unbounded gains from betting and quasi-coherence

As in Section 3, we restrict attention to the case where N = {1,2}.

Let $\mathcal{U}$ be the set of differentiable functions u: $\mathbb{R} \to \mathbb{R}$ that satisfy the following properties:

(1) u is increasing, concave, u(0) = 0, u′(0) = 1, and

(2) $\lim_{x\to+\infty} u(x) = +\infty$ and $\lim_{x\to+\infty} u'(x) = 0$.

$\mathcal{U}$ will be referred to as the set of *moderately risk-averse* utility functions.

DEFINITION 7. A *moderately risk-averse betting environment based on the frame* $\mathcal{F}$ is a tuple $\mathcal{E} = \langle \mathcal{B}, \{u_i\}_{i\in N} \rangle$ where $\mathcal{B}$ is a Bayesian frame based on $\mathcal{F}$ and, for every i ∈ N, $u_i : \Omega \to \mathcal{U}$ is a function that associates with every state a utility function in the class $\mathcal{U}$ satisfying the property that, $\forall\ \alpha,\beta \in \Omega$, if $p_{i,\beta} = p_{i,\alpha}$ then $u_{i,\alpha} = u_{i,\beta}$, that is, each individual knows her own utility function.

DEFINITION 8. Within a risk-averse betting environment, a *proposed bet* is a vector $x \in \mathbb{R}^{|N|\,|\Omega|}$ such that, for every $\omega \in \Omega$, $\sum_{i\in N} x_{i,\omega} = 0$; $x_{i,\omega}$ is the payment (to, if positive, or from, if negative) individual i at state ω.

Given an individual i, a state α and a proposed bet x, let

$$v_i(\alpha, x) = \sum_{\omega\in\Omega} u_{i,\alpha}(x_{i,\omega})\ p_{i,\alpha}(\omega)$$

Thus $v_i(\alpha, x)$ is individual i's expected utility at state α (given her utility function and beliefs at that state) if she accepts the proposed bet x. Given a number $\xi \in \mathbb{R}$, let $\|v_i(x) \geq \xi\|$ denote the event that individual i's expected utility (if she accepts the proposed trade x) is at least ξ:

$\|v_i(x) \geq \xi\| = \{ \alpha \in \Omega : v_i(\alpha, x) \geq \xi \}$.

DEFINITION 9. A risk-averse betting environment admits *unbounded gains from betting* at α if and only if

$$\forall\, \xi \in \mathbb{R}^{+},\ \exists\ x \in \mathbb{R}^{|N|\,|\Omega|} \quad \text{such that } \alpha \in B_*\Big(\bigcap_{i \in N} \|v_i(x) \geq \xi\|\Big).$$

that is, if, for every positive number ξ, there is a proposed bet x such that at α it is common belief that everybody has an expected utility of at least ξ if she accepts the bet x.

Quasi-coherence is equivalent to the absence of unbounded gains from betting,

PROPOSITION 4. Let $\mathcal{F}$ be a frame where N = {1, 2} and α a state. Then $\mathcal{F}$ admits unbounded gains from betting at α if and only if $\alpha \in \neg\mathbf{Q}$.

REMARK 5. The proof of Proposition 4 makes essential use of the notion of moderate risk-aversion. For the necessity part the crucial property is unboundedness of the utility function, while the sufficiency part relies on the property that marginal utility tends to zero as x tends to infinity. Without the assumption that $\lim_{x \to +\infty} u'(x) = 0$ it is possible to have unbounded gains from betting at α even if $\alpha \in B_*\mathbf{T}$ [14].

REMARK 6. In the case of complete information (where each individual has only one "type", that is, the beliefs of every individual are commonly known) $\mathbf{Q} = B_*\mathbf{T}$. Hence common belief in no error is equivalent to the impossibility of unbounded gains from betting and, by Proposition 3, to qualitative agreement.

5. Decomposing the Truth Axiom

In this section we allow for any number of individuals. To explore the gap between quasi-coherence and common belief in no error we introduce the following events ($\mathbf{T}_{CB}$

stands for Truth *about* common belief, while $\mathbf{T}^*$ stands for Truth *of* common belief):

$$\mathbf{T}_{CB} = \bigcap_{i \in N} \bigcap_{E \in 2^{\Omega}} \neg(B_i B_* E \cap \neg B_* E)$$

$$\mathbf{T}^* = \bigcap_{E \in 2^{\Omega}} \neg(B_* E \cap \neg E) .$$

$\mathbf{T}_{CB}$ captures the notion that individuals are correct in their beliefs about what is commonly believed: $\alpha \in \mathbf{T}_{CB}$ if and only if, for every event E and individual i, if, at α, individual i believes that E is commonly believed, then, at α, E is indeed commonly believed (if $\alpha \in B_i B_* E$ then $\alpha \in B_* E$). On the other hand, $\alpha \in \mathbf{T}^*$ if and only if at α whatever is commonly believed is true (for every event E, if $\alpha \in B_* E$ then $\alpha \in E$). Clearly, Truth of common belief is qualitatively weaker than Truth; given that $B_* \mathbf{T}^* = \Omega$, $\mathbf{T}^*$ can be viewed as Truth shorn of any intersubjective implications. It is straightforward that $\alpha \in \mathbf{T}^*$ if and only if, $\alpha \in I_*(\alpha)$. To check whether $\mathbf{T}_{CB}$ holds at a state, the following lemma (proved in Bonanno and Nehring, 1997) is useful.

LEMMA 2. $\alpha \in \mathbf{T}_{CB}$ if and only if $\forall i \in N$, $\forall \beta \in I_*(\alpha)$, $\exists \delta \in I_i(\alpha)$ such that $\beta \in I_*(\delta)$.

Lemma 2 says that at α Truth about common belief holds if and only if whenever it is possible to go from α to β with the common possibility correspondence then it must be possible to go from α to β in two steps using the possibility correspondence of an arbitrary individual for the first step and the common possibility correspondence for the second step. Thus, for example, in Figure 1 $\gamma \notin \mathbf{T}_{CB} = \{\tau, \beta\}$, since $\gamma \in I_*(\gamma)$ and $I_1(\gamma) = \{\beta\}$ and γ is not commonly reachable from β: $I_*(\beta) = \{\beta\}$. In fact, at γ individual 1 erroneously believes that it is common belief that it is cloudy: $\gamma \in B_1 B_* E \cap \neg B_* E$ where $E = \{\beta\}$.

The following proposition gives a decomposition of the Truth Axiom in terms of quasi-coherence, Truth of common belief and (common belief in) Truth about common belief.

PROPOSITION 5. $\mathbf{T} \cap B_*\mathbf{T} = \mathbf{T}^* \cap B_*\mathbf{T}_{CB} \cap \mathbf{Q}$.

REMARK 7. Since $B_*\mathbf{T}^* = \Omega$, it follows that $B_*\mathbf{T} = B_*(\mathbf{T}_{CB} \cap \mathbf{Q})$

REMARK 8. The events $\mathbf{T}^*$, $B_*\mathbf{T}_{CB}$ and $\mathbf{Q}$ are pairwise independent, that is, in general, none of them is a subset of the other. This can be seen as follows:

For the pair ($\mathbf{T}^*$, $B_*\mathbf{T}_{CB}$): in Figure 1, $\mathbf{T}^* = \Omega \not\subseteq B_*\mathbf{T}_{CB} = \{\beta\}$[15]; in the following frame: $N = \{1,2\}$, $\Omega = \{\tau, \beta\}$, $I_i(\omega) = \{\beta\}$ for all $i \in N$ and $\omega \in \Omega$, $B_*\mathbf{T}_{CB} = \Omega \not\subseteq \mathbf{T}^* = \{\beta\}$.

For the pair ($\mathbf{Q}$, $\mathbf{T}^*$): in the frame just described, $\mathbf{Q} = \Omega \not\subseteq \mathbf{T}^* = \{\beta\}$; in Figure 3, $\mathbf{T}^* = \Omega \not\subseteq \mathbf{Q} = \varnothing$.

For the pair ($\mathbf{Q}$, $B_*\mathbf{T}_{CB}$): in Figure 3, $B_*\mathbf{T}_{CB} = \Omega \not\subseteq \mathbf{Q} = \varnothing$; in the following frame: $N = \{1,2\}$, $\Omega = \{\tau, \beta\}$, $I_1(\tau) = I_1(\beta) = \{\beta\}$, $I_2(\tau) = \{\tau\}$, $I_2(\beta) = \{\beta\}$,

$\mathbf{Q} = \Omega \not\subseteq B_*\mathbf{T}_{CB} = \{\beta\}$.

REMARK 9. None of $\mathbf{T}^*$, $\mathbf{T}_{CB}$ and $B_*\mathbf{T}_{CB}$, either individually or in conjunction with the others, has any "agreement" implications. This can be seen from Figure 4 where $\mathbf{T}^* = \mathbf{T}_{CB} = B_*\mathbf{T}_{CB} = \Omega$ and yet at both τ and β the individuals agree to strongly disagree: $B_*(B_1\neg E \cap B_2E) = \Omega$ where $E = \{\tau\}$. On the other hand, by Propositions 1 and 4, $\mathbf{Q}$ is precisely the property that captures the notion of agreement.

Insert Figure 4

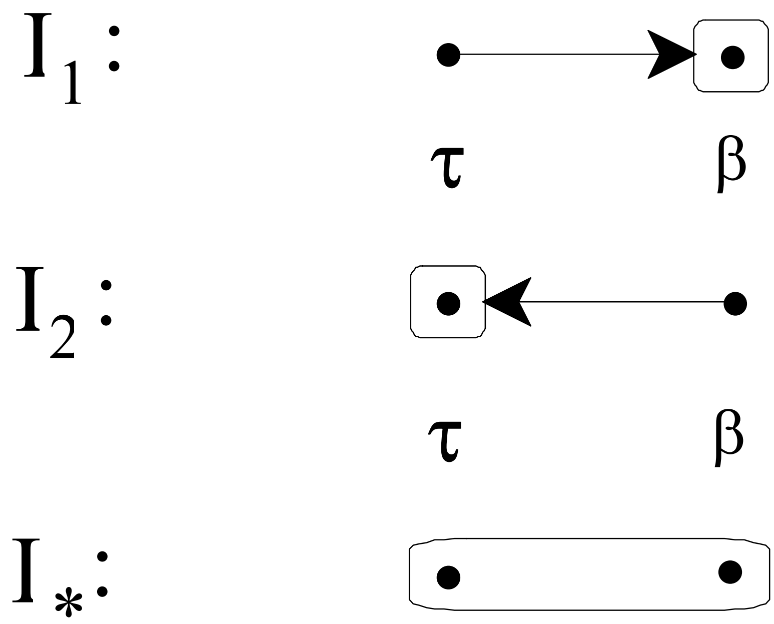

Figure 4

$\mathbf{T}^*$, $\mathbf{T}_{CB}$ and $B_*\mathbf{T}_{CB}$ do not imply "agreement"

6. Conclusion

The Truth Condition (or the somewhat weaker hypothesis of common belief in no error) plays an important role in the foundations of backward induction (Ben Porath, 1997, Stalnaker, 1994, 1996, Stuart, 1997). In Sections 3 and 4 it was shown that the content of $B_*\mathbf{T}$ can only in part be reduced to the intersubjective notions of agreement and bounded gains from betting. From an economic point of view the absence of unbounded gains from betting seems a very compelling assumption. While economic agents engage in (explicit or implicit) bets with each other (e.g. in the stock market), the associated gains from trade seem far less than infinite.[16] However, it can be shown (Bonanno and Nehring, 1997, Stuart, 1997) that the property that characterizes the absence of unbounded gains from betting (quasi coherence) is by itself too weak to yield the desired implications in game theoretic reasoning. By Proposition 5, in order to restore the game-theoretic implications of the Truth Condition one needs to augment quasi coherence with the properties of Truth of common belief and Truth about common belief. Thus, given the plausibility of the assumption of quasi-coherence, the burden of justifying the Truth Condition lies in the latter assumptions. However, as noted in Remark 9 and illustrated in Figures 3 and 4, both $\mathbf{T}^*$ and $B_*\mathbf{T}_{CB}$ are by themselves significantly weaker than the Truth Condition. Thus Proposition 5 can be viewed as providing support for the latter's plausibility.

The above defense of the Truth Condition based on the absence of unbounded gains from trade is sensitive to a Bayesian definition of certain beliefs. Within more general models of decision making under uncertainty, absence of unbounded gains from trade no longer implies quasi-coherence[17] since an individual's failure to be certain of some event E no longer necessarily entails a willingness to bet *against E* even at extremely favorable odds.

This is illustrated by the following example, in which individuals are assumed to maximize the minimum expected utility of an act relative to a set of "admissible priors" (for an axiomatic foundation of this MMEU model, see Gilboa-Schmeidler, 1989; the example is also consistent with the Choquet-Expected Utility Model, or with Bewley's, 1986, inertia model).

DEFINITION 10. A multi-prior frame is a tuple $\mathcal{M} = \langle N, \Omega, \tau, \{\Pi_i\}_{i\in N} \rangle$

where, as before, N is the set of individuals, Ω is a finite set of states, τ is the true state and

- for every individual $i \in N$, $\Pi_i : \Omega \to 2^{\Delta(\Omega)}$ is a function that associates with every state a set of probability distributions on Ω satisfying the following property, which says that individual i knows her own beliefs:

$$\forall\, \alpha,\beta \in \Omega, \text{ if } p(\beta) > 0 \text{ for some } p \in \Pi_{i,\alpha} \text{ then } \Pi_{i,\alpha} = \Pi_{i,\beta} \qquad (7)$$

(A Bayesian frame as defined in Section 3 is thus a special case of this where, for every i and α, $\Pi_{i,\alpha}$ is a singleton.)

It has been observed before in the literature that in a non-additive context, there is more than one (at least minimally) plausible way to define certainty in terms of preferences; see, e.g. Morris (1995) and Sarin and Wakker (1995). Both of these papers advocate a definition of the certainty of an event E as equivalent to its complement being a Savage-null event. A distinctive advantage of this definition is that it ensures that the conjunction of certain events is itself a certain event. Its privileged status is further confirmed by the analysis of Nehring (1997) which proposes a definition of unambiguous probabilistic beliefs based on preferences. Specialized to unambiguous probability-one beliefs, this definition coincides with the suggested one.

In the MMEU model, the suggested definition of "individual i is certain that E" is equivalent to requiring that *all* of i's admissible priors assign probability one to E. This leads to the following definition of the corresponding belief-operator.

DEFINITION 11. Given a multi-prior frame, the belief operator of individual i, $B_i : 2^{\Omega} \to 2^{\Omega}$, is defined by:

$$\alpha \in B_i E \text{ if and only if, } \forall\, p \in \Pi_{i,\alpha},\ p(E) = 1.$$

REMARK 10. It is easily verified that the belief operator thus defined is normal and satisfies consistency as well as positive and negative introspection (cf. Remark 3). It is also straightforward that the associated possibility correspondence $I_i : \Omega \to 2^{\Omega}$ is given by $I_i(\alpha) = \bigcup_{p \in \Pi_{i,\alpha}} \text{supp}(p)$.

EXAMPLE 1. Consider the following multi-prior frame: N = {1,2}, $\Omega = \{\tau, \beta, \gamma\}$, $\Pi_{1,\tau} = \Pi_{1,\delta} = \Pi_{1,\gamma} = \{p\}$ with $p(\tau) = p(\delta) = \frac{1}{2}$, $\Pi_{2,\tau} = \{q\}$ with $q(\tau) = 1$, $\Pi_{2,\delta} = \Pi_{2,\gamma} =$ $\{p \in \Delta(\Omega) : p(\tau) = 0 \text{ and } p(\gamma) \leq 10^{-6}\}$. Note that the corresponding qualitative frame (obtained by replacing the function Π_i with the possibility correspondence defined in Remark 10) is the one illustrated in Figure 3. In particular, the agents' beliefs fail to be quasi-coherent at τ.

The notion of unbounded gains from trade generalizes naturally to multi-prior frames by redefining $v_i(\alpha, x)$ (see Section 4, after Definition 8) as follows:

$$v_i(\alpha, x) = \min_{p \in \Pi_{i,\alpha}} \sum_{\omega \in \Omega} u_{i,\alpha}(x_{i,\omega}) p(\omega)$$

It is easily verified that, despite the failure of quasi-coherence, the frame of Example 1 does *not* admit unbounded (or indeed any) gains from betting. The reason for this is as follows. For x to be a bet with the property that at τ there is common belief that both individuals derive positive expected utility from accepting x , it is necessary that $x_{2,\tau}$ (the payment to individual 2 at τ) be positive; hence, since $x_{1,\tau} < 0$, $x_{1,\delta}$ must be positive, i.e. $x_{2,\delta}$ must be negative. However, $x_{2,\delta} < 0$ implies that the minimum expected utility of individual 2 at δ (and γ) is negative, since one $p \in \Pi_{2,\delta}$ assigns probability 1 to δ. The essential feature of this example is the failure of the admissible priors of individual 2 at β and γ to have the same support.

Clearly, the Bayesian analysis of Section 4 generalizes to multi-prior frames in which agents' beliefs are commonly known to have this property. While not implausible, there seems to be nothing particularly "necessary" about it; indeed, it seems quite sensible for an individual to recognize the non-trivial possibility of an event (i.e. assign *upper* probability greater than zero to it), while abstaining from putting any strictly positive confidence in its occurrence (i.e. while assigning *lower* probability zero to it).

The above discussion confirms the implicit premise of this paper that the assumption of common belief in no error ($B_* T$) is not as simple as it may seem, in that its interpretation and justification depend on the meaning attached to the notion of "belief" or "certainty". Understanding common belief in no error in the context of more general interpretations of certain beliefs is an issue that deserves further exploration.

Appendix

REMARK A.1. A possibility correspondence $I : \Omega \to 2^{\Omega}$ is *secondary reflexive* if $\forall\, \alpha, \beta \in \Omega$, $\beta \in I(\alpha)$ implies $\beta \in I(\beta)$. Secondary reflexivity is implied by euclideanness. Hence, for every $i \in N$, I_i is secondary reflexive. It follows from the definition of I_* that I_* is also secondary reflexive.

Proof of Observation 1. ($B_i\mathbf{T_j} \subseteq \mathbf{L_{ij}}$) Let $\alpha \in B_i\mathbf{T_j}$. Fix an arbitrary event E. Then $\alpha \in B_i(\neg B_jE \cup E)$. We want to show that $\alpha \in \neg B_iB_jE \cup B_iE$. Suppose that $\alpha \in B_iB_jE$. Then $\alpha \in B_iB_jE \cap B_i(\neg B_jE \cup E) =_{\text{(by Conjunction)}} B_i(B_jE \cap (\neg B_jE \cup E)) = B_i\,(B_jE \cap E) \subseteq_{\text{(by Monotonicity)}} B_iE$.

($\mathbf{L_{ij}} \subseteq B_i\mathbf{T_j}$) Let $\alpha \in \mathbf{L_{ij}}$. Then $\alpha \in \neg B_iB_j\mathbf{T_j} \cup B_i\mathbf{T_j}$. Since $B_j\mathbf{T_j} = \Omega$ (it follows from secondary reflexivity of I_j), $B_iB_j\mathbf{T_j} = \Omega$ (by Necessity). Hence $\alpha \in B_iB_j\mathbf{T_j}$. Thus $\alpha \in B_i\mathbf{T_j}$.

Since $B_i\mathbf{T_j} = \mathbf{L_{ij}}$, by Conjunction $\mathbf{L} = \bigcap_{i \in N} B_i\mathbf{T}$. Hence $B_*\mathbf{L} = B_*\mathbf{T}$. ■

Proof of Proposition 2. We only need to show that $\mathcal{F}_Q \cap \mathcal{F}_2 \subseteq \mathcal{F}_S$. Fix an arbitrary $f \in \mathcal{F}_Q \cap \mathcal{F}_2$ and let the random variable $Y : \Omega \to \mathbb{R}$ represent f according to Proposition 1. If $Y(\omega) \geq 0$ for all $\omega \in \Omega$, then $f = f_{\Omega} \in \mathcal{F}_S$. Otherwise, there exists $\omega \in \Omega$ such that $Y(\omega) < 0$. Then, since $f \in \mathcal{F}_Q$, it must be $Y(\omega) \leq 0$ for all. This implies that $f = f_E \in \mathcal{F}_S$ where $E = \{\omega \in \Omega : Y(\omega) = 0\}$. ■

Proof of Lemma 1. Call P the property stated in Lemma 1. Assume P. Choose arbitrary $E \subseteq \Omega$, $i, j \in \{1,2\}$. By Conjunction, $\neg B_*(B_iE \cap \neg B_jE) = \neg B_*B_iE \cup \neg B_*\neg B_jE$. Fix an arbitrary $\alpha \in B_*B_iE$. We want to show that $\alpha \in \neg B_*\neg B_jE$, that is, $I_*(\alpha) \cap B_jE \neq \emptyset$. By P there exists a $\delta \in I_*(\alpha)$ such that $I_j(\delta) \subseteq \bigcup_{\omega \in I_*(\alpha)} I_i(\omega)$. Since $\alpha \in B_*B_iE$, $I_*(\alpha) \subseteq B_iE$. Thus $\forall\, \omega \in I_*(\alpha)$, $I_i(\omega) \subseteq E$.

Hence $\bigcup_{\omega \in I_*(\alpha)} I_i(\omega) \subseteq E$. Hence $I_j(\delta) \subseteq E$, that is, $\delta \in B_jE$.

Assume that P is violated. Then there exist $i,j \in N$ and $\alpha \in \Omega$ such that, for all $\delta \in I_*(\alpha)$,

$I_j(\delta) \nsubseteq \bigcup_{\omega \in I_*(\alpha)} I_i(\omega)$. Let $E = \bigcup_{\omega \in I_*(\alpha)} I_i(\omega)$. Then, for all $\omega \in I_*(\alpha)$, $I_i(\omega) \subseteq E$, that is, $\omega \in B_iE$. Thus

$I_*(\alpha) \subseteq B_iE$, i.e. $\alpha \in B_*B_iE$. On the other hand, $I_*(\alpha) \cap B_jE = \varnothing$, because if $\delta \in I_*(\alpha) \cap B_jE$ then $I_j(\delta) \subseteq E$, contradicting our hypothesis. Hence $\alpha \notin \neg B_*\neg B_jE$. Thus $\alpha \in B_*(B_iE \cap \neg B_jE)$. ■

LEMMA A.1. $\forall\, \alpha \in \Omega$, $\alpha \in B_*\mathbf{T}$ if and only if, $\forall\; i \in N$, $\{I_i(\omega) : \omega \in I_*(\alpha)\}$ is a partition of $I_*(\alpha)$ [note, however, that it is possible that $\alpha \notin I_*(\alpha)$].

Proof. By euclideanness and transitivity of I_i, for every $\beta, \gamma \in \Omega$, either $I_i(\beta) = I_i(\gamma)$ or $I_i(\beta) \cap I_i(\gamma) = \varnothing$. Thus the stated condition is equivalent to

$$\forall\; i \in N, \quad \bigcup_{\omega \in I_*(\alpha)} I_i(\omega) = I_*(\alpha) \tag{A.1}$$

Fix an arbitrary $i \in N$. First note that

$$\forall\, \alpha \in \Omega, \quad \bigcup_{\omega \in I_*(\alpha)} I_i(\omega) \subseteq I_*(\alpha) \tag{A.2}$$

In fact, by definition of I_*, for every $\omega \in \Omega$, $I_i(\omega) \subseteq I_*(\omega)$. Hence $\bigcup_{\omega \in I_*(\alpha)} I_i(\omega) \subseteq \bigcup_{\omega \in I_*(\alpha)} I_*(\omega)$. By transitivity of I_*, if $\omega \in I_*(\alpha)$ then $I_*(\omega) \subseteq I_*(\alpha)$. Hence $\bigcup_{\omega \in I_*(\alpha)} I_*(\omega) \subseteq I_*(\alpha)$. Thus we only need to show that $\alpha \in B_*\mathbf{T}$ if and only if,

$$\forall\, i \in N, \quad \bigcup_{\omega \in I_*(\alpha)} I_i(\omega) \supseteq I_*(\alpha) \tag{A.3}.$$

Let $\alpha \in B_*\mathbf{T}$. Then

$$\forall\, \beta \in I_*(\alpha), \quad \beta \in \bigcap_{i \in N} I_i(\beta). \tag{A.4}$$

Fix an arbitrary $\beta \in I_*(\alpha)$. By (A.4), for all i ∈ N, $\beta \in I_i(\beta) \subseteq \bigcup_{\omega \in I_*(\alpha)} I_i(\omega)$. Conversely, assume (A.3). Fix an arbitrary $\beta \in I_*(\alpha)$ and an arbitrary i ∈ N. Then, by (A.3), $\beta \in \bigcup_{\omega \in I_*(\alpha)} I_i(\omega)$. Hence there exists a $\gamma \in I_*(\alpha)$ such that $\beta \in I_i(\gamma)$. By secondary reflexivity of I_i (cf. Remark A.1), $\beta \in I_i(\beta)$. Thus (A.4) is satisfied and hence $\alpha \in B_*\mathbf{T}$. ■

Proof of Corollary 1. Fix an arbitrary $\alpha \in \mathbf{Q}$. Then there exists a $\beta \in I_*(\alpha)$ such that $\beta \in B_*\mathbf{T}$. By secondary reflexivity of I_* (cf. Remark A.1), $\beta \in I_*(\beta)$. By Lemma A.1, $\bigcup_{\omega \in I_*(\beta)} I_1(\omega) = \bigcup_{\omega \in I_*(\beta)} I_2(\omega)$. Thus for every i,j∈N, $I_i(\beta) \subseteq \bigcup_{\omega \in I_*(\beta)} I_i(\omega) = \bigcup_{\omega \in I_*(\beta)} I_j(\omega)$. Hence, by Lemma 1, $\alpha \in \mathcal{F}_S$-**Agree**. ■

LEMMA A.2. Let $S \subseteq \Omega$ be such that, for all j ∈ N, $\bigcup_{s \in S} I_j(s) \subseteq S$. If, for all i,j∈ N, $\bigcup_{s \in S} I_i(s) = \bigcup_{s \in S} I_j(s)$ then, for all j ∈ N, $\bigcup_{s \in S} I_j(s) = \bigcup_{s \in S} I_*(s)$.

Proof. Fix an arbitrary j∈N and an arbitrary s∈S. By definition of I_*, $I_j(s) \subseteq I_*(s)$. Hence $\bigcup_{s \in S} I_j(s) \subseteq \bigcup_{s \in S} I_*(s)$. Next we show that $\bigcup_{s \in S} I_j(s) \supseteq \bigcup_{s \in S} I_*(s)$. Choose an arbitrary $\gamma \in \bigcup_{s \in S} I_*(s)$. Then there exists a $\beta \in S$ such that $\gamma \in I_*(\beta)$. By definition of I_*, there exist a sequence $\langle \delta_0, ..., \delta_m \rangle$ in Ω and a sequence $\langle k_1, ..., k_m \rangle$ in N such that $\delta_0 = \beta$, $\delta_m = \gamma$ and for every r = 1, ..., m, $\delta_r \in I_{k_r}(\delta_{r-1})$. Since, by hypothesis, for every individual k∈N, $\bigcup_{s \in S} I_k(s) \subseteq S$ and $\beta \in S$, it follows that the sequence $\langle \delta_0, ..., \delta_m \rangle$ is entirely in S. Moreover $\gamma = \delta_m \in I_{k_m}(\delta_{m-1})$. By secondary reflexivity of I_{k_m} (cf. Remark A.1), $\gamma \in I_{k_m}(\gamma)$. Since $\gamma \in S$, $I_{k_m}(\gamma) \subseteq \bigcup_{s \in S} I_{k_m}(s)$. By hypothesis, $\bigcup_{s \in S} I_j(s) = \bigcup_{s \in S} I_{k_m}(s)$. Hence $\gamma \in \bigcup_{s \in S} I_j(s)$. ■

COROLLARY A.1. Let $\alpha,\beta \in \Omega$ be such that $\beta \in I_*(\alpha)$ and $\forall$ i , j $\in$ N,

$\bigcup_{\omega\in I_*(\beta)} I_i(\omega) = \bigcup_{\omega\in I_*(\beta)} I_j(\omega)$. Then, for all i $\in$ N, $\bigcup_{\omega\in I_*(\beta)} I_i(\omega) = I_*(\beta)$.

Proof. By definition of I_*, for every $\omega \in \Omega$ and for every i $\in$ N, $I_i(\omega) \subseteq I_*(\omega)$. Thus $\bigcup_{\omega\in I_*(\beta)} I_i(\omega) \subseteq \bigcup_{\omega\in I_*(\beta)} I_*(\omega)$. Hence, by Lemma A.2 [with S = $\bigcup_{\omega\in I_*(\beta)} I_*(\omega)$], for all i $\in$ N, $\bigcup_{\omega\in I_*(\beta)} I_i(\omega) = \bigcup_{\omega\in I_*(\beta)} I_*(\omega)$. By secondary reflexivity of I_* (cf. Remark A.1), since $\beta \in I_*(\alpha)$, $\beta \in I_*(\beta)$. Thus $\bigcup_{\omega\in I_*(\beta)} I_*(\omega) \supseteq I_*(\beta)$. In conjunction with (A.2) this yields $\bigcup_{\omega\in I_*(\beta)} I_*(\omega) = I_*(\beta)$. ■

LEMMA A.3. Consider a frame where N = {1,2}. Fix an arbitrary $\alpha \in \Omega$. Then the following are equivalent:

(i) For all $\mathcal{T}_1 \subseteq \{ I_1(\omega) : \omega \in I_*(\alpha) \}$ and $\mathcal{T}_2 \subseteq \{ I_2(\omega) : \omega \in I_*(\alpha) \}$, $\bigcup\mathcal{T}_1 \neq \bigcup\mathcal{T}_2$.

(ii) $\forall \beta \in I_*(\alpha)$, $\bigcup_{\omega\in I_*(\beta)} I_1(\omega) \neq \bigcup_{\omega\in I_*(\beta)} I_2(\omega)$.

Proof. (i) $\Rightarrow$ (ii). Choose an arbitrary $\beta \in I_*(\alpha)$. By transitivity of I_*, $I_*(\beta) \subseteq I_*(\alpha)$. Choose $\mathcal{T}_1 = \{ I_1(\omega) : \omega \in I_*(\beta) \}$ and $\mathcal{T}_2 = \{ I_2(\omega) : \omega \in I_*(\beta) \}$. Then, by (i), $\bigcup_{\omega\in I_*(\beta)} I_1(\omega) = \bigcup\mathcal{T}_1 \neq \bigcup\mathcal{T}_2 = \bigcup_{\omega\in I_*(\beta)} I_2(\omega)$.

not (i) $\Rightarrow$ not (ii). Suppose that (i) is violated, that is, there exist $\mathcal{T}_1 \subseteq \{ I_1(\omega) : \omega \in I_*(\alpha) \}$ and $\mathcal{T}_2 \subseteq \{ I_2(\omega) : \omega \in I_*(\alpha) \}$ such that $\bigcup\mathcal{T}_1 = \bigcup\mathcal{T}_2$. Let T = $\bigcup\mathcal{T}_1 = \bigcup\mathcal{T}_2$. By transitivity of I_*, T $\subseteq I_*(\alpha)$. By transitivity and euclideanness of I_i, the elements of $\{ I_i(\omega) : \omega \in I_*(\alpha) \}$ are pairwise disjoint. Hence $\mathcal{T}_1$ and $\mathcal{T}_2$ are partitions of T. Thus $\bigcup_{\omega\in T} I_1(\omega) = \bigcup_{\omega\in T} I_2(\omega)$ = T. Then, by Lemma A.2, $\bigcup_{\omega\in T} I_*(\omega)$ = T. Fix an arbitrary $\beta \in$ T. Then $I_*(\beta) \subseteq$ T. By (ii), $\bigcup_{\omega\in I_*(\beta)} I_1(\omega) \neq \bigcup_{\omega\in I_*(\beta)} I_2(\omega)$. Then there is an i$\in$N and a $\gamma \in \Omega$ such that $\gamma \in \bigcup_{\omega\in I_*(\beta)} I_i(\omega) - \bigcup_{\omega\in I_*(\beta)} I_j(\omega)$ where

$j \in N-\{i\}$. Since $\gamma \in \bigcup_{\omega \in I_*(\beta)} I_i(\omega) \subseteq \bigcup_{\omega \in T} I_i(\omega)$ (because $I_*(\beta) \subseteq T$) and $\bigcup_{\omega \in T} I_1(\omega) = \bigcup_{\omega \in T} I_2(\omega)$,

there exists a $\delta \in T$ such that $\gamma \in I_j(\delta)$. Then $\gamma \in I_j(\gamma)$ by secondary reflexivity of I_j. Since

$\gamma \in \bigcup_{\omega \in I_*(\beta)} I_i(\omega)$, there exists a $\theta \in I_*(\beta)$ such that $\gamma \in I_i(\theta)$. Then, by definition of I_*,

$\gamma \in I_*(\beta)$. Thus $\gamma \in I_j(\gamma) \subseteq \bigcup_{\omega \in I_*(\beta)} I_j(\omega)$, contradicting our hypothesis that

$\gamma \in \bigcup_{\omega \in I_*(\beta)} I_i(\omega) - \bigcup_{\omega \in I_*(\beta)} I_j(\omega)$. ■

COROLLARY A.2. Consider a frame where $N = \{1,2\}$ and $\alpha \in \neg\mathbf{Q}$. Then
$\forall \beta \in I_*(\alpha)$, $\bigcup_{\omega \in I_*(\beta)} I_1(\omega) \neq \bigcup_{\omega \in I_*(\beta)} I_2(\omega)$.

Proof. Suppose that for some $\beta \in I_*(\alpha)$, $\bigcup_{\omega \in I_*(\beta)} I_1(\omega) = \bigcup_{\omega \in I_*(\beta)} I_2(\omega)$. Then, by Corollary A.1,

$\bigcup_{\omega \in I_*(\beta)} I_1(\omega) = \bigcup_{\omega \in I_*(\beta)} I_2(\omega) = I_*(\beta)$. Then by (A.3) $\beta \in B_*\mathbf{T}$. Hence $\alpha \in \neg B_*\neg B_*\mathbf{T} = \mathbf{Q}$. ■

Proof of Proposition 3. Since $\mathcal{F}_Q$**-Agree** $\supseteq \mathcal{F}_S$**-Agree**, by Corollary 1, $\mathcal{F}_Q$**-Agree** $\supseteq \mathbf{Q}$. Thus it only remains to show that $\mathcal{F}_Q$**-Agree** $\subseteq \mathbf{Q}$ or, equivalently, that $\neg\mathbf{Q} \subseteq \neg\mathcal{F}_Q$**-Agree**. Let $\alpha \in \mathbf{Q}$. By Corollary A.2,

$$\forall \beta \in I_*(\alpha),\ \bigcup_{\omega \in I_*(\beta)} I_1(\omega) \neq \bigcup_{\omega \in I_*(\beta)} I_2(\omega). \tag{A.5}$$

The crucial step of the construction is to ensure that the belief index to be defined is proper, that is (cf. Remark 4), satisfies union consistency. The key to this is Lemma A.3 above. Thus, for every $i \in N$, let $\mathcal{A}_i$ be the closure under union of the set $\{ I_i(\omega) : \omega \in I_*(\alpha)\}$. By (A.5) and Lemma A.3,

$$\mathcal{A}_1 \cap \mathcal{A}_2 = \varnothing. \tag{A.6}$$

Define the following function $f : \Delta(\Omega) \rightarrow 2^{\Omega} \cup \{0,1\}$: $f(p) = \begin{cases} 1 & \text{if supp(p)} \in \mathcal{A}_1 \\ 0 & \text{if supp(p)} \in \mathcal{A}_2 \\ \text{supp(p)} & \text{otherwise} \end{cases}$.

Then, by (A.6) f is well-defined and, furthermore, $f \in \mathcal{F}_Q$; in particular, f is proper. By definition of $\mathcal{A}_i$, $\forall\ \omega \in I_*(\alpha)$, $f(p_{1,\omega}) = 1$ and $f(p_{2,\omega}) = 0$. Thus $\alpha \in B_*(\| f_1 = 1\| \cap \| f_2 = 0\|)$. Hence $\alpha \notin f$ **-Agree.** ■

LEMMA A.4. Fix a (not necessarily moderately) risk-averse betting environment. Choose an arbitrary individual i and an arbitrary state α. Let m be the number of elements in $I_i(\alpha)$ and $p_0 = \min\{p_{i,\alpha}(\omega) : \omega \in I_i(\alpha)\}$. By definition of Bayesian model, m > 0 and $p_0 > 0$. Let $\xi > 0$ be such that

$$\frac{u_{i,\alpha}(\xi)}{\xi} \le \frac{p_0}{m} \qquad \text{(A.7)}.$$

Then, for every proposed bet x,

$$\text{if} \quad v_i(\alpha, x) \equiv \sum_{\omega\in\Omega} u_{i,\alpha}(x_{i,\omega})\ p_{i,\alpha}(\omega) \ge \xi \quad \text{then} \quad \sum_{\omega\in I_i(\alpha)} \tfrac{1}{m} x_{i,\omega} > 0 \qquad \text{(A.8)}.$$

Proof. First note that every function $u \in \mathcal{U}$ satisfies the following properties [because of concavity of u together with the normalization u′(0) = 1]:

$$\forall\, y \in \mathbb{R}, \quad u(y) \le y, \quad \text{and} \qquad \text{(A.9)}$$

$$\forall\, y,y' \in \mathbb{R}, \quad y' \ge y \Rightarrow \frac{u(y')}{y'} \le \frac{u(y)}{y} \qquad \text{(A.10)}$$

By definition of $I_i(\alpha)$, $\sum_{\omega\in\Omega} u_{i,\alpha}(x_{i,\omega})\ p_{i,\alpha}(\omega) = \sum_{\omega\in I_i(\alpha)} u_{i,\alpha}(x_{i,\omega})\ p_{i,\alpha}(\omega)$. Hence, by (A.9), $v_i(\alpha, x) \le \sum_{\omega\in I_i(\alpha)} x_{i,\omega}\ p_{i,\alpha}(\omega)$. Assume that $v_i(\alpha, x) \ge \xi > 0$. Then $x_{i,\beta} \ge \xi$ for some $\beta \in I_i(\alpha)$. Let

$y^{max} = \max \{x_{i,\omega} : \omega \in I_i(\alpha)\}$. Thus $y^{max} \geq \xi$. Hence, by (A.10) and the hypothesis that

$\frac{u_{i,\alpha}(\xi)}{\xi} \leq \frac{p_0}{m}$

$$\frac{u_{i,\alpha}(y^{max})}{y^{max}} \leq \frac{p_0}{m}. \tag{A.11}$$

It follows from (A.11) (recall that $y^{max} \geq \xi > 0$) that

$$-\frac{u_{i,\alpha}(y^{max})}{p_0} \geq -\frac{y^{max}}{m}. \tag{A.12}$$

Let $y_{min} = \min \{x_{i,\omega} : \omega \in I_i(\alpha)\}$. Now, $0 < \xi \leq \sum_{\omega \in I_i(\alpha)} u_{i,\alpha}(x_{i,\omega})\ p_{i,\alpha}(\omega) \leq p_0\, u_{i,\alpha}(y_{min}) + u_{i,\alpha}(y^{max})$.

Hence

$$u_{i,\alpha}(y_{min}) > -\frac{u_{i,\alpha}(y^{max})}{p_0} \qquad \text{(A.13).}$$

By (A.9),

$$y_{min} \geq u_{i,\alpha}(y_{min}) \qquad \text{(A.14).}$$

It follows from (A.12)-(A.14) that

$$y_{min} > -\frac{y^{max}}{m} \tag{A.15}$$

Hence $\sum_{\omega \in I_i(\alpha)} \frac{1}{m} x_{i,\omega} \geq \frac{m-1}{m}\, y_{min} + \frac{1}{m}\, y^{max} > \frac{m-1}{m}\left(-\frac{y^{max}}{m}\right) + \frac{1}{m} y^{max} = \frac{1}{m^2} y^{max} > 0$. ■

Proof of Proposition 4. Proposition 4 follows from the following:

(i) If there exists a moderately risk-averse betting environment based on $\mathcal{F}$ that admits unbounded gains from betting at α, then $\alpha \in \neg\mathbf{Q}$, and

(ii) if $\alpha \in \neg\mathbf{Q}$, then every moderately risk-averse every betting environment based on $\mathcal{F}$ admits unbounded gains from betting at α.

Proof of (i). Fix a moderately risk-averse betting environment that admits unbounded gains from betting at state α, that is,

$$\forall\, \xi \in \mathbb{R}^{+},\ \exists\ x \in \mathbb{R}^{|N|\,|\Omega|} \text{ such that } \alpha \in B_*\Big(\bigcap_{i\in N} \|v_i(x) \geq \xi\|\Big). \tag{A.16}$$

Choose a ξ large enough that (A.7) is satisfied for all $i\in N$ and all $\beta\in I_*(\alpha)$ (this is possible because both N and Ω are finite and, by definition of moderately risk-averse utility function u, $\lim_{x\to+\infty} u'(x) = 0$). Then, by (A.16) and Lemma A.4 there exists a proposed bet x such that, $\forall$ $i\in N$ and $\forall\, \beta\in I_*(\alpha)$, $\frac{1}{|I_i(\beta)|}\sum_{\omega\in I_i(\beta)} x_{i,\omega} > 0$ (where $|I_i(\beta)|$ denotes the cardinality of $I_i(\beta)$). Hence

$$\forall\ i\in N \text{ and } \forall\ \beta\in I_*(\alpha), \qquad \sum_{\omega\in I_i(\beta)} x_{i,\omega} > 0 \tag{A.17}$$

Suppose that $\alpha \in \neg B_*\neg B_*\mathbf{T}$. Then there exists an $\gamma \in I_*(\alpha)$ such that $\gamma \in B_*\mathbf{T}$. By Lemma A.1, for every $i \in N$, $\{I_i(\omega) : \omega \in I_*(\gamma)\}$ is a partition of $I_*(\gamma)$. By transitivity of I_*, $I_*(\gamma) \subseteq I_*(\alpha)$. Thus, by (A.17),

$$\forall\ i\in N, \qquad \sum_{\omega\in I_*(\gamma)} x_{i,\omega} > 0 \tag{A.18}$$

It follows from (A.18) that $\sum_{i\in N}\sum_{\omega\in I_*(\gamma)} x_{i,\omega} > 0$. But $\sum_{i\in N}\sum_{\omega\in I_*(\gamma)} x_{i,\omega} = \sum_{\omega\in I_*(\gamma)}\sum_{i\in N} x_{i,\omega} = 0$ by definition of proposed bet, yielding a contradiction.

Proof of (ii) Suppose that $\alpha \in \neg\mathbf{Q}$. Fix an arbitrary risk-averse betting environment based on this frame. Let $\mathcal{I}_1 = \bigcup_{\omega\in I_*(\alpha)} I_1(\omega) - \bigcup_{\omega\in I_*(\alpha)} I_2(\omega)$ and $\mathcal{I}_2 = \bigcup_{\omega\in I_*(\alpha)} I_2(\omega) - \bigcup_{\omega\in I_*(\alpha)} I_1(\omega)$. By Lemma A.3 (choosing $\mathcal{T}_i = \{\, I_i(\omega) : \omega \in I_*(\alpha) \,\}$), $(\mathcal{I}_1 - \mathcal{I}_2) \cup (\mathcal{I}_2 - \mathcal{I}_1) \neq \varnothing$. Fix an arbitrary $\hat{\beta} \in (\mathcal{I}_1 - \mathcal{I}_2) \cup (\mathcal{I}_2 - \mathcal{I}_1)$. Suppose, without loss of generality, that $\hat{\beta} \in \mathcal{I}_1 - \mathcal{I}_2$. For every $i \in N$ define $\delta^i_{\hat{\beta}} : \Omega \to \mathbb{N} \cup \{\infty\}$ as follows (see Figure 5): $\delta^i_{\hat{\beta}}(\omega) = \infty$ if $\hat{\beta} \notin \bigcup_{\omega'\in I_i(\omega)} I_*(\omega')$ (that is, if $\hat{\beta}$ cannot be reached from ω with a sequence of "steps" in which the first step is taken by individual i) and otherwise

$\delta^i_{\hat\beta}(\omega) = \ell$ where ℓ is the *smallest* positive integer such that there exists a sequence $\langle \omega_k \rangle_{k=0,1,\ldots,\ell}$

satisfying the following conditions: (1) $\omega_0 = \omega$, (2) $\omega_\ell = \hat\beta$, and (iii) for all k = 0, ..., ℓ, if k+1 is odd then $\omega_{k+1} \in I_i(\omega_k)$ and if k+1 is even then $\omega_{k+1} \in I_j(\omega_k)$ with j ≠ i. Note that, by construction, $\forall\, \omega,\omega' \in \Omega$, $\forall$ i∈N,

if $I_i(\omega) = I_i(\omega')$ then $\delta^i_{\hat\beta}(\omega) = \delta^i_{\hat\beta}(\omega')$, $\delta^1_{\hat\beta}(\omega)$ is odd (if finite) and $\delta^2_{\hat\beta}(\omega)$ is even (if finite) (A.19)

Finally, define $\delta_{\hat\beta} : \Omega \to \mathbb{N} \cup \{\infty\}$ as follows: $\delta_{\hat\beta}(\hat\beta) = 0$, and, for every $\omega \in \Omega \setminus \{\hat\beta\}$, $\delta_{\hat\beta}(\omega) =$ min $\{\delta^1_{\hat\beta}(\omega), \delta^2_{\hat\beta}(\omega)\}$. Note that, by construction,

$\forall$ i∈N, $\forall\, \omega,\omega' \in \Omega$ with $\delta_{\hat\beta}(\omega) < \infty$, $I_i(\omega) = I_i(\omega')$ and $\omega \in I_i(\omega)$, $|\delta_{\hat\beta}(\omega') - \delta_{\hat\beta}(\omega)| \leq 1$. (A.20)

Furthermore,

$\forall\ \gamma,\eta \in \Omega$, if $\eta \in I_1(\gamma)$ and $\delta_{\hat\beta}(\eta)$ is odd then $\exists\, \theta \in I_1(\gamma)$ such that $\delta_{\hat\beta}(\theta) = \delta_{\hat\beta}(\eta) - 1$ (A.21)

In fact, if $\delta_{\hat\beta}(\eta) = 1$ [i.e. $\hat\beta \in I_1(\eta)$] then $\theta = \hat\beta$, since, by transitivity and euclideanness of I_1, $I_1(\eta)$ $= I_1(\gamma)$. Suppose $\delta_{\hat\beta}(\eta) = \ell > 1$. Then there exists a sequence $\langle \omega_k \rangle_{k=0,\ldots,\ell}$ such that $\omega_0 = \eta$, $\omega_\ell = \hat\beta$ and for all k = 0, ..., ℓ, if k+1 is odd then $\omega_{k+1} \in I_1(\omega_k)$ and if k+1 is even then $\omega_{k+1} \in I_2(\omega_k)$. Thus $\omega_1 \in I_1(\eta)$, $\delta^1_{\hat\beta}(\omega_1) = \delta^1_{\hat\beta}(\eta)$ and $\delta^2_{\hat\beta}(\omega_1) = \ell - 1$. Hence $\delta_{\hat\beta}(\omega_1) = \ell - 1$ and we can choose $\theta = \omega_1$, since, by transitivity and euclideanness of I_1, $I_1(\eta) = I_1(\omega_1)$. Similarly,

$\forall\ \gamma,\eta \in \Omega$, if $\eta \in I_2(\gamma)$ and $\delta_{\hat\beta}(\eta)$ is even then $\exists\, \theta \in I_2(\gamma)$ such that $\delta_{\hat\beta}(\theta) = \delta_{\hat\beta}(\eta) - 1$ (A.22)

We want to show that for every $\xi > 0$ there exists a bet x such that it is common belief at α that every individual has an expected utility of at least ξ. The bet we will construct is such that $x_{1,\omega} < 0$

(hence $x_{2,\omega} > 0$) if and only if $\delta_{\hat{\beta}}(\omega)$ is odd and $x_{1,\omega} > 0$ (hence $x_{2,\omega} < 0$) if and only if $\delta_{\hat{\beta}}(\omega)$ is even. The following figure (where α is the true state τ) illustrates the idea of the proof.

Insert Figure 5

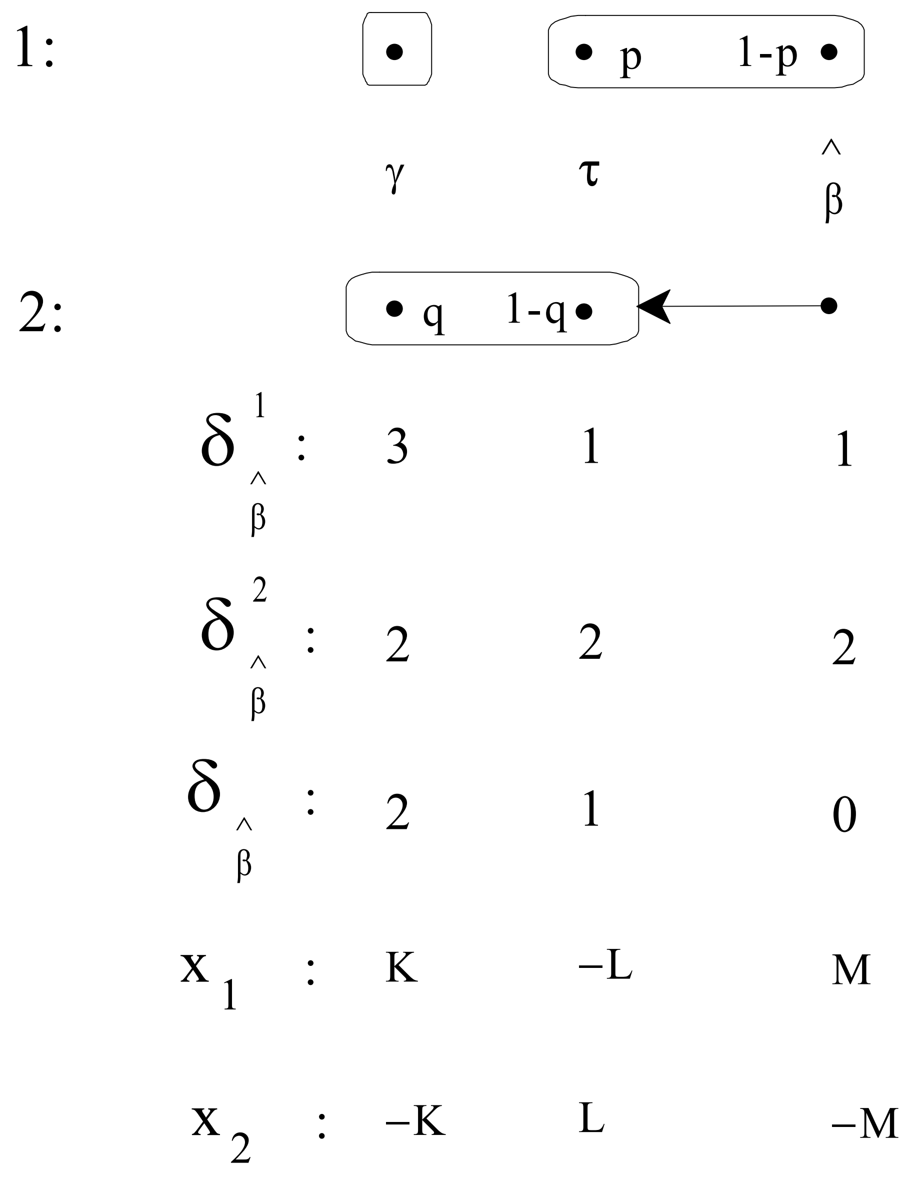

Figure 5

The construction in the proof of Proposition 4

In Figure 5 the numbers K, L and M are all positive and chosen to satisfy the following conditions: (i) $u_{1,\gamma}(K) \geq \xi$, (ii) $q\, u_{2,\tau}(-K) + (1-q)\, u_{2,\tau}(L) \geq \xi$, (iii) $p\, u_{1,\tau}(-L) + (1-p)\, u_{1,\tau}(M) \geq \xi$. These conditions can be satisfies, because of the assumption that, for every utility function $u \in \mathcal{U}$, $\lim_{x\to+\infty} u(x) = +\infty$.

Restrict attention to bets x that satisfy the following properties:

(i) if $\delta_{\hat{\beta}}(\omega)$ is odd then $x_{1,\omega} < 0$ and $\forall\, \omega' \in I_i(\omega)$ if $\delta_{\hat{\beta}}(\omega') = \delta_{\hat{\beta}}(\omega)$ then $x_{1,\omega'} = x_{1,\omega}$.

(ii) if $\delta_{\hat{\beta}}(\omega)$ is even then $x_{1,\omega} > 0$ and $\forall\, \omega' \in I_i(\omega)$ if $\delta_{\hat{\beta}}(\omega') = \delta_{\hat{\beta}}(\omega)$ then $x_{1,\omega'} = x_{1,\omega}$.

Claim 1. Fix an arbitrary $\gamma \in \Omega$ such that $\delta_{\hat{\beta}}(\gamma) < \infty$ and an arbitrary $\xi > 0$. Then there exists a bet satisfying (i) and (ii) above such that

$$\text{(1)} \qquad \text{if } \gamma\in I_1(\gamma) \text{ [i.e. } p_{1,\gamma}(\gamma) > 0] \text{ then } \sum_{\omega\in I_1(\gamma)} p_{1,\gamma}(\omega)\; u_{1,\gamma}(x_{1,\omega}) \geq \xi \text{ and}$$

$$\text{(2)} \qquad \text{if } \gamma\in I_2(\gamma) \text{ [i.e. } p_{2,\gamma}(\gamma) > 0] \text{ then } \sum_{\omega\in I_2(\gamma)} p_{2,\gamma}(\omega)\; u_{2,\gamma}(x_{2,\omega}) \geq \xi. \tag{A.23}$$

Proof of claim. Fix an arbitrary $\xi > 0$ and an arbitrary $\gamma \in I_*(\alpha)$ such that $\gamma \in \bigcup_{\omega\in I_1(\alpha)} I_*(\omega)$ and $\gamma\in I_1(\gamma)$. If $\delta_{\hat{\beta}}(\omega)$ is even for every $\omega \in I_1(\gamma)$, then $x_{1,\omega} = K$ for some $K > 0$ for all $\omega \in I_1(\gamma)$ and hence $\sum_{\omega\in I_1(\gamma)} p_{1,\gamma}(\omega)\; u_{1,\gamma}(x_{1,\omega}) = u_{1,\gamma}(K)$ which can be made to exceed ξ by choosing K large enough. Suppose now that $\delta_{\hat{\beta}}(\eta)$ is odd for some $\eta \in I_1(\gamma)$. Then by (A.21) there exists a $\theta \in I_1(\gamma)$ such that $\delta_{\hat{\beta}}(\theta) = \delta_{\hat{\beta}}(\eta) - 1$ and by (A.20) for every $\omega \in I_1(\gamma)$ either $\delta_{\hat{\beta}}(\omega) = \delta_{\hat{\beta}}(\eta)$ or $\delta_{\hat{\beta}}(\omega) = \delta_{\hat{\beta}}(\eta) - 1$. Fix a number ε such that $0 < \varepsilon \leq \min_{i\in\{1,2\},\, \omega\in I_*(\alpha)} \{p_{i,\omega}(\omega):\; p_{i,\omega}(\omega) > 0\}$. Then $\sum_{\omega\in I_1(\gamma)} p_{1,\gamma}(\omega)\; u_{1,\gamma}(x_{1,\omega}) \geq (1-\varepsilon)\, u_{1,\gamma}(x_{1,\eta}) + \varepsilon\, u_{1,\gamma}(x_{1,\theta})$ (recall that $x_{1,\eta} < 0$ and $x_{1,\theta} > 0$). Thus, given any $x_{1,\eta}$ by choosing $x_{1,\theta}$ large enough it is possible to make this expression exceed ξ. The proof of (2) is similar. To ensure the simultaneous satisfaction of (1) and (2) for the same γ, note first that by Lemma A.3 and Corollary A.2, $I_1(\gamma) \neq I_2(\gamma)$. Let $\omega \in I_1(\gamma) \cup I_2(\gamma)$ be a minimizer of $\delta_{\hat{\beta}}(\cdot)$ over $I_1(\gamma) \cup I_2(\gamma)$. Let i be the individual for whom $\omega \in I_i(\gamma)$. Then $x_{i,\omega} > 0$. Thus one can first fix the payment from i to j ($j \neq i$) at states $\omega' \in I_j(\gamma)$ such that $\delta_{\hat{\beta}}(\omega') > \delta_{\hat{\beta}}(\omega)$ so as to ensure an expected utility of at least ξ to individual j at γ and then choose $x_{i,\omega}$ large enough so that individual i's expected utility at γ is also greater than ξ. [end of proof of Claim]

Now, two cases are possible: (1) $I_*(\alpha) = \bigcup_{\omega \in I_1(\alpha)} I_*(\omega) = \bigcup_{\omega \in I_2(\alpha)} I_*(\omega)$ and (2) $I_*(\alpha) \neq \bigcup_{\omega \in I_i(\alpha)} I_*(\omega)$ for exactly one $i \in N$. Consider first case (1). Fix $\xi > 0$. By (A.23), starting from the state that maximizes $\delta_{\hat{\beta}}(\cdot)$ over $I_*(\alpha)$ and proceeding inductively towards $\hat{\beta}$ one can ensure an expected utility of at least ξ to both individuals at every $\omega \in I_*(\alpha)$, that is, a bet x can be constructed such that $\alpha \in B_*(\bigcap_{i \in N} \|v_i(x) \geq \xi\|)$. In case (2), let i be the individual for whom $\bigcup_{\omega' \in I_i(\alpha)} I_*(\omega) \neq I_*(\alpha)$.

Then choose a $\hat{\gamma} \in \big((I_1 - I_2) \cup (I_2 - I_1)\big) \cap \big(I_*(\alpha) \setminus \bigcup_{\omega \in I_i(\tau)} I_*(\omega)\big)$ and repeat the construction above to obtain bets $y_{i,\omega}$ [with the sign reversed if $\hat{\gamma} \in I_2(\hat{\gamma})$]. Define $z_{i,\omega} = y_{i,\omega} + x_{i,\omega}$. Then, by repeating the argument based on (A.22), one can show that $\alpha \in B_*(\bigcap_{i \in N} \|v_i(z) \geq \xi\|)$. ■

DEFINITION A.1. Let $\mathbf{NI}^*$ (for Negative Introspection of common belief) be the following event:

$$\mathbf{NI}^* = \bigcap_{E \in 2^{\Omega}} (B_*E \cup B_*\neg B_*E)$$

Thus $\alpha \in \mathbf{NI}^*$ if and only if, for every event E, if, at α, E is not commonly believed then, at α, it is commonly believed that E is not common belief.

REMARK A.2. It is well-known that, $\forall\, \alpha \in \Omega$, $\alpha \in \mathbf{NI}^*$ if and only if $I_*(\alpha)$ is euclidean, that is, $\beta \in I_*(\alpha) \Rightarrow I_*(\alpha) \subseteq I_*(\beta)$.

LEMMA A.5. $\forall\, \alpha \in \Omega$, if $\alpha \in \mathbf{T} \cap B_*\mathbf{T}$ then $\forall\, \beta \in I_*(\alpha)$, $I_*(\alpha) = I_*(\beta)$. It follows (see Remark A.2) that $\mathbf{T} \cap B_*\mathbf{T} \subseteq \mathbf{NI}^*$.[18]

Proof. That $I_*(\beta) \subseteq I_*(\alpha)$ follows from transitivity of I_*. Thus we only need to show that $I_*(\alpha) \subseteq I_*(\beta)$ [that is, euclideanness of $I_*(\alpha)$]. By definition of I_*, since $\beta \in I_*(\alpha)$, there exists a sequence $\langle j_1, \ldots j_m \rangle$ in N and a sequence $\langle \eta_0, \eta_1, \ldots, \eta_m \rangle$ in Ω such that: $\eta_0 = \alpha$, $\eta_m = \beta$ and, for every k = 0, ..., m − 1, $\eta_{k+1} \in I_{j_{k+1}}(\eta_k)$. By transitivity and euclideanness of I_{j_1}, since $\eta_1 \in I_{j_1}(\alpha)$, $I_{j_1}(\alpha) = I_{j_1}(\eta_1)$. Since $\alpha \in \mathbf{T}$, $\alpha \in I_{j_1}(\alpha)$. Thus $\alpha \in I_{j_1}(\eta_1)$. Since $\eta_1 \in I_*(\alpha)$ and $\alpha \in B_*\mathbf{T}$, $\eta_1 \in I_{j_2}(\eta_1)$. Since $\eta_2 \in I_{j_2}(\eta_1)$, by transitivity and euclideanness of I_{j_2}, $I_{j_2}(\eta_1) = I_{j_2}(\eta_2)$. Thus $\eta_1 \in I_{j_2}(\eta_2)$. Repeating this argument m times, we get that, $\forall k = 0, \ldots, m$, $\eta_k \in I_{j_{k+1}}(\eta_{k+1})$. Thus, by definition of I_*, $\eta_0 \in I_*(\eta_m)$, that is, $\alpha \in I_*(\beta)$. It follows from transitivity of I_* that $I_*(\alpha) \subseteq I_*(\beta)$. ■

LEMMA A.6. For every event $E \subseteq \Omega$, $B_*B_*E = B_*E$.

Proof. It is well known (see Chellas, 1984, pp. 164 and 92) that, for *any* belief operator B, BBE = BE for every event E if the corresponding possibility correspondence is transitive and secondary reflexive. I_* indeed satisfies both properties (cf. Remark A.1). ■

PROPOSITION A.1 (Bonanno and Nehring, 1997). $\mathbf{NI}^* = \mathbf{T}_{CB} \cap B_*\mathbf{T}_{CB}$.

LEMMA A.7. (i) $\mathbf{Q} \cap \mathbf{T}_{CB} \cap B_*\mathbf{T}_{CB} \subseteq B_*\mathbf{T}$, (ii) $B_*\mathbf{T} \subseteq B_*\mathbf{T}_{CB}$.

Proof. (i) First we show that

$$\neg B_*\neg B_*\mathbf{T} \cap \mathbf{NI}^* \subseteq B_*\mathbf{T}. \quad \text{(A.24)}$$

Let $\alpha \in \neg B_*\neg B_*\mathbf{T} \cap \mathbf{NI}^*$. Since $\alpha \in \mathbf{NI}^*$, $\alpha \in B_*\mathbf{T} \cup B_*\neg B_*\mathbf{T}$. Hence, since $\alpha \in \neg B_*\neg B_*\mathbf{T}$,

$\alpha \in B_*\mathbf{T}$.

(ii) By Lemma A.5, $\mathbf{T} \cap B_*\mathbf{T} \subseteq \mathbf{NI}^*$. Hence by Monotonicity of B_*, $B_*\mathbf{T} \cap B_*B_*\mathbf{T} \subseteq B_*\mathbf{NI}^*$. By Lemma A.6, $B_*B_*\mathbf{T} = B_*\mathbf{T}$ and by Proposition A.1, $B_*\mathbf{NI}^* = B_*\mathbf{T}_{CB} \cap B_*B_*\mathbf{T}_{CB}$. By Lemma A.6, $B_*B_*\mathbf{T}_{CB} = B_*\mathbf{T}_{CB}$. ■

Proof of Proposition 5. By definition of $\mathbf{T}$ and $\mathbf{T}^*$,

$$\mathbf{T} \subseteq \mathbf{T}^* \qquad \text{(A.25).}$$

By (ii) of Lemma A.7,

$$B_*\mathbf{T} \subseteq B_*\mathbf{T}_{CB} \qquad \text{(A.26).}$$

By Lemma A.6, $B_*\mathbf{T} = B_*B_*\mathbf{T}$ and by non-empty-valuedness of I_*, $B_*B_*\mathbf{T} \subseteq \neg B_*\neg B_*\mathbf{T}$. Thus

$$B_*\mathbf{T} \subseteq \neg B_*\neg B_*\mathbf{T} \qquad \text{(A.27).}$$

Thus, by (A.25)-(A.27), $\mathbf{T} \cap B_*\mathbf{T} \subseteq \mathbf{T}^* \cap B_*\mathbf{T}_{CB} \cap \neg B_*\neg B_*\mathbf{T}$.

By definition of $\mathbf{T}^*$, $\mathbf{T}^* \cap B_*\mathbf{T}_{CB} \subseteq \mathbf{T}_{CB}$. Thus $\mathbf{T}^* \cap B_*\mathbf{T}_{CB} = \mathbf{T}^* \cap \mathbf{T}_{CB} \cap B_*\mathbf{T}_{CB}$. By Proposition A.1, $\mathbf{T}^* \cap \mathbf{T}_{CB} \cap B_*\mathbf{T}_{CB} = \mathbf{T}^* \cap \mathbf{NI}^*$. Thus

$$\mathbf{T}^* \cap B_*\mathbf{T}_{CB} = \mathbf{T}^* \cap \mathbf{NI}^*. \qquad \text{(A.28).}$$

It follows from (A.28) that

$$\mathbf{T}^* \cap B_*\mathbf{T}_{CB} \cap \neg B_*\neg B_*\mathbf{T} = \mathbf{T}^* \cap \mathbf{NI}^* \cap \neg B_*\neg B_*\mathbf{T}. \qquad \text{(A.29).}$$

By (A.24), $\mathbf{T}^* \cap \mathbf{NI}^* \cap \neg B_*\neg B_*\mathbf{T} \subseteq \mathbf{T}^* \cap B_*\mathbf{T}$. By definition of $\mathbf{T}^*$, $\mathbf{T}^* \cap B_*\mathbf{T} \subseteq \mathbf{T}$. Thus $\mathbf{T}^* \cap B_*\mathbf{T}_{CB} \cap \neg B_*\neg B_*\mathbf{T} \subseteq \mathbf{T} \cap B_*\mathbf{T}$. ■

References

R. Aumann, Agreeing to disagree, *Annals of Statistics* 4 (1976) 1236-1239.

R. Aumann, Correlated equilibrium as an expression of Bayesian rationality, *Econometrica* 55 (1987) 1-18.

R. Aumann, Backward induction and common knowledge of rationality, *Games and Economic Behavior* 88 (1995) 6-19.

R. Aumann, Deriving backward induction in the centipede game without assuming rationality at unreached vertices, mimeo, The Hebrew University, Jerusalem, 1996.

R. Aumann and A. Brandenburger, Epistemic conditions for Nash equilibrium, *Econometrica* 63 (1995) 1161-1180

M. Bacharach, Some extensions of a claim of Aumann in an axiomatic model of knowledge, *Journal of Economic Theory* 37 (1985) 167-190.

E. Ben-Porath, Rationality, Nash equilibrium and backward induction in perfect information games, *Review of Economic Studies* 64 (1997) 23-46.

T. Bewley, Knightian decision theory, Part I, Cowles Foundation Discussion Paper No. 807, Yale University, 1986.

G. Bonanno, On the logic of common belief, *Mathematical Logic Quarterly* 42 (1996) 305-311.

G. Bonanno and K. Nehring, Making sense of the common prior assumption under incomplete information, Working Paper, University of California, Davis, 1996.

G. Bonanno and K. Nehring, On the logic and role of negative introspection of common belief, Working Paper, University of California, Davis, 1997 [forthcoming in *Mathematical Social Sciences*].

B. Chellas, *Modal logic* (Cambridge University Press, Cambridge, 1984).

E. Dekel and F. Gul, Rationality and knowledge in game theory, in: D. M. Kreps and K. F. Wallis, eds., *Advances in Economics and Econometrics*, vol. 1 (Cambridge University Press, Cambridge, 1997) pp. 87-172.

R. Fagin, J. Halpern, Y. Moses and M. Vardi, *Reasoning about knowledge* (MIT Press, Cambridge, 1995).

J. Geanakoplos, Common knowledge, *Journal of Economic Perspectives* 6 (1992) 53-82.

I. Gilboa, and D. Schmeidler, Maximin expected utility with a non-unique prior, *Journal of Mathematical Economics* 18 (1989) 141-153.

J. Halpern, and Y. Moses, A guide to completeness and complexity for modal logics of knowledge and belief, *Artificial intelligence* 54 (1992) 319-379.

L. Lismont and P. Mongin, On the logic of common belief and common knowledge, *Theory and Decision* 37 (1994) 75-106.

L. Lismont and P. Mongin, Belief closure: a semantics for common knowledge for modal propositional logic, *Mathematical Social Sciences* 30 (1995) 127-153.

S. Morris, Trade with heterogeneous prior beliefs and asymmetric information, *Econometrica* 62 (1994) 1327-1347.

S. Morris, Alternative notions of belief, mimeo, University of Pennsylvania, 1995 [forthcoming in: L. Gerard-Varet, P. Mongin and H. Shin, eds., *Epistemic logic and the theory of games and decisions*, Kluwer].

Y. Moses and G. Nachum, Agreeing to disagree after all, in: R. Parikh, ed., *Theoretical aspects of reasoning about knowledge: proceedings of the third conference* (Morgan Kaufmann, San Francisco, 1990), pp. 151-168.

K. Nehring, Capacities and probabilistic beliefs: a precarious coexistence, mimeo, University of California, Davis, 1997 [forthcoming in *Mathematical Social Sciences*].

A. Rubinstein and A. Wolinsky, On the logic of “Agreeing to Disagree” type of results, *Journal of Economic Theory* 51 (1990) 184-193.

R. Sarin and P. Wakker, On the interpretation of likelihood in Choquet expected utility, mimeo, University of Leiden, 1995.

R. Stalnaker, On the evaluation of solution concepts, *Theory and Decision* 37 (1994) 49-74.

R. Stalnaker, Knowledge, belief and counterfactual reasoning in games, *Economics and Philosophy* 12 (1996) 133-163.

H. Stuart, Common belief of rationality in the finitely repeated Prisoners’ Dilemma, *Games and Economic Behavior* 19 (1997) 133-143.

FOOTNOTES

[1] See, for example, Aumann (1976, 1987, 1995, 1996), Bacharach (1985), Geanakoplos (1992), Fagin *et al* (1995).

[2] Implicitly we are referring to common belief in no error as "coherence of beliefs".

[3] We have included the true state in the definition of an interactive belief frame in order to stress the interpretation of the frame as a representation of a particular profile of hierarchies of beliefs.

[4] See, for example, Aumann (1987), Aumann and Brandenburger (1995), Dekel and Gul (1997), Morris (1994), Stalnaker (1994, 1996).

[5] A directed graph is *asymmetric* if, whenever there is an arrow from vertex v to vertex v′ then there is no arrow from v′ to v.

[6] See, for example, Bonanno (1996), Fagin et al (1995), Halpern and Moses (1992), Lismont and Mongin (1994, 1995). These authors also show that the common belief operator can be alternatively defined by means of a finite list of axioms, rather than as an infinite conjunction.

[7] It is well known that $\alpha \in \mathbf{T_j}$ if and only if $\alpha \in I_j(\alpha)$.

[8] Bacharach (1985) generalized Aumann's result to the non-Bayesian case of "decision functions" that satisfy the "Sure Thing Principle". Roughly speaking he showed that if two individuals are "like-minded" (in the sense that they choose their actions based on a common decision procedure that satisfies the sure thing principle) and reach common knowledge of the actions each of them intends to perform, then they will perform identical actions. Like Aumann, Bacharach assumes the S5 logic for individual beliefs (information partitions). Moses and Nachum (1990) show that assuming like-mindedness and the sure thing principle

amounts to assuming much more than the principle that, given the same information, the two individuals would behave in the same way and raise doubts as to whether it is a meaningful assumption at all.

[9] If μ is a probability distribution over Ω, we denote by supp(μ) the support of μ, that is, the set of states to which μ assigns positive probability.

[10] The following are examples of proper belief indices:

(i) Let $E \subseteq \Omega$ be an arbitrary event, $X = [0,1]$ and f^E the following belief index: $f^E(p) = p(E) \equiv \sum_{\omega \in E} p(\omega)$; thus, $f^E(p_{i,\alpha})$ is individual i's subjective probability of event E at state α.

(ii) Let $Y : \Omega \to \mathbb{R}$ be a random variable, $X = \mathbb{R}$ and f_Y be the belief index given by

$f_Y(p) = \sum_{\omega \in \Omega} Y(\omega) p(\omega)$; thus $f_Y(p_{i,\alpha})$ is i's subjective expectation of Y at state α.

(iii) Let A be a set of actions, $X = 2^A$ and $U : A \times \Omega \to \mathbb{R}$ a utility function. Define the belief index $f_U : \Delta(\Omega) \to 2^A$ as follows: $f_U(p) = \arg\max_{a \in A} \sum_{\omega \in \Omega} U(a,\omega)\, p(\omega)$. Thus $f_U(p_{i,\alpha})$ is the set of actions that maximize individual i's expected utility at state α.

[11] Relative to the standard Euclidean topology (recall that Ω is assumed to be finite).

[12] To represent f_E in the manner of Proposition 1, let $Y : \Omega \to \mathbb{R}$ be as follows: $Y = 1_E - 1$, where $1_E : \Omega \to \{0,1\}$ is the indicator function of E: $1_E(\omega) = 1$ if and only if $\omega \in E$. Hence

$Y(\omega) = \begin{cases} 0 & \text{if } \omega \in E \\ -1 & \text{if } \omega \notin E \end{cases}$. Then $\sum_{\omega \in \Omega} Y(\omega)p(\omega) = \sum_{\omega \in \neg E} Y(\omega)p(\omega) = -\sum_{\omega \in \neg E} p(\omega) < 0$ if and only if $p(\omega) > 0$ for some $\omega \in \neg E$, if and only if $\sum_{\omega \in E} p(\omega) < 1$.

[13] Lemma 1 states that $\alpha \in \mathcal{I}_S$-**Agree** if and only if for every individual i there exists a state β commonly accessible from α satisfying the following property: if state γ is considered possible by j at β then γ is

considered possible by i at some $\omega \in I_*(\alpha)$. For example, in Figure 3 the above property is satisfied at every state (for both 1 and 2 the role of β is played by τ).

[14] For example, let N = {1,2}, $\Omega = \{\tau, \beta\}$, $\forall$ i$\in$N and $\forall\ \omega\in\Omega$, $I_i(\omega) = \Omega$, $p_{1,\tau}(\tau) = p_{2,\tau}(\beta) = 0.8$, $p_{1,\tau}(\beta) = p_{2,\tau}(\tau) = 0.2$, $u_{1,\tau} = u_{2,\tau} = u$ with u(x) = x if x ≥ 0 and u(x) = 2x if x < 0. Then $B_*\mathbf{T} = \Omega$ and at τ (and β) there are unbounded gains from betting (the bet $x_{1,\tau} = x_{2,\beta} = \xi > 0$, $x_{1,\beta} = x_{2,\tau} = -\xi$ yields an expected utility to each individual of $0.4\,\xi$, which can be made arbitrarily large by increasing ξ).

[15] Since $\mathbf{T}_{CB} = \{\beta, \tau\}$ and $I_*(\tau) = \{\beta, \gamma, \tau\}$, $\tau \notin B_*\mathbf{T}_{CB}$.

[16] I.e. far below the restrictions implied by the agents' wealth constraints.

[17] Nor is it necessarily implied by it, depending on the model.

[18] Note that, without euclideanness, for a single individual it is not true that the conjunction of correct beliefs and belief in correct beliefs yields negative introspection (even in the presence of positive introspection). The proof of Lemma A.5 relies in an essential way on euclideanness of I_i for every individual i.