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The Logical Representation of Extensive Games

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Abstract: Given an extensive form G, we associate with every choice an atomic sentence and with every information set a set of well-formed formulas (wffs) of propositional calculus. The set of such wffs is denoted by $\Gamma(G)$. Using the so-called topological semantics for propositional calculus (which differs from the standard one based on truth tables), we show that the extensive form yields a topological model of $\Gamma(G)$, that is, every wff in $\Gamma(G)$, is "true in G". We also show that, within the standard truth-table semantics for propositional calculus, there is a one-to-one and onto correspondence between the set of plays of G and the set of valuations that satisfy all the wffs in $\Gamma(G)$.

1 Introduction

A number of recent papers[2] have been concerned with applying the methods of logic to the analsis of games. A pre-condition for this is that there be a rigorous and unambiguous way of translating any given game into a set of well-formed formulas of propositional calculus. Consider, for example, the extensive form with perfect information shown in Figure 1[3]. Let A_{I} stand for the proposition "player I selects choice a", C_{II} stand for the proposition "payer II selects choice c", etc. Then one could use the following propositions to describe the game of Figure 1.

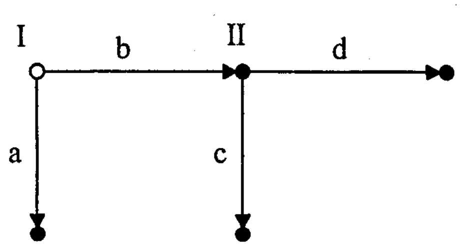

Fig. 1.

[1] I am very grateful to two anonymous referees for their detailed and constructive comments on the first version of this paper.

[2] See, for example, Anderlini (1990), Bacharach (1987), Bonanno (1991), Canning (1992), Kaneko and Nagashima (1991), Kramarz (1992), Vilks (1992).

[3] The precise definition of extensive form will be given later (in section 3, for the case of perfect information, and in section 4, for the general case).

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(1) $A_{\rm I} \vee B_{\rm I}$ ("either player I selects choice *a* or she selects choice *b*"),
(2) $\neg(A_{\rm I} \wedge B_{\rm I})$ ("it is not the case that player I selects both choice *a* and choice *b*"),
(3) $(C_{\rm II} \vee D_{\rm II}) \Leftrightarrow B_{\rm I}$ ("either player II selects choice *c* or he selects choice *D*, if and only if player I selects choice *b*"),
(4) $\neg(C_{\rm II} \wedge D_{\rm II})$ ("it is not the case that player II selects both choice *c* and choice *d*").

Although the above propositions seem to be intuitively acceptable as a logical description of the extensive form of Figure 1, there are a number of questions that can be raised:

(i) Is there a precise sense in which propositions (1)–(4) are true for the extensive form of Figure 1?
(ii) What does it mean for a player to "select" a choice? Does it mean that the player actually **makes** that choice or could it mean that the player **plans** to make that choice? If the latter is the case, then proposition (3) does not seem to be necessarily true: player II can plan to choose *c* even if player I does not choose *b*[4]. Thus maybe proposition (3) should be replaced by

   (3b) $(C_{\rm II} \vee D_{\rm II})$.

   Which of the two (3 or 3b) is true for the game of Figure 1?
(iii) Can we be sure that – whatever method we use to translate extensive forms into a set of propositions – two "essentially different" extensive forms will have different logical representations? This important point was raised in a recent note by Arnis Vilks (1992b) (see also Bonanno, 1992a). He gave the following example, reproduced (with slight modifications) in Figure 2.

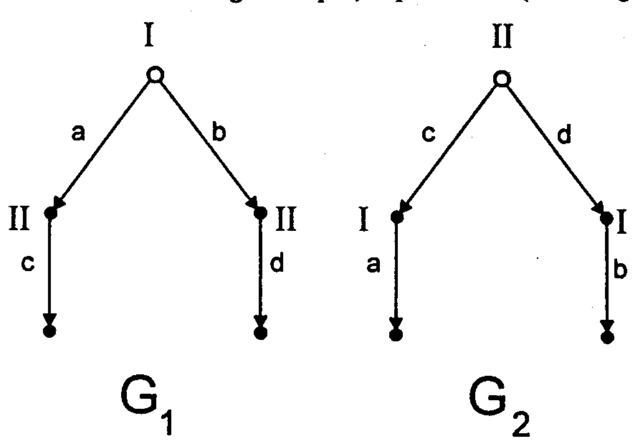

Fig. 2.

Vilks argues that (using the same symbolism as before) extensive form G_1 has the following propositional representation:

$$(A_{\rm I} \vee B_{\rm I}) \wedge \neg(A_{\rm I} \wedge B_{\rm I}) \wedge (A_I \Leftrightarrow C_{\rm II}) \wedge (B_{\rm I} \Leftrightarrow D_{\rm II})$$

[4] Kramarz (1992) puts forward the point of view that (3) should be replaced by (3b).

while extensive form G_2 has the following propositional representation:

$$(C_{II} \vee D_{II}) \wedge \neg(C_{II} \wedge D_{II}) \wedge (C_{II} \Leftrightarrow A_I) \wedge (D_{II} \Leftrightarrow B_I).$$

It is easy to check that the two formulas are logically equivalent (one implies the other). Thus – Vilks argues – the two extensive forms have the same propositional representation. However, a suitable choice of payoffs may make "*ac*" the rational play of G_1 and "*db*" the rational play of G_2, so that G_1 and G_2 can be viewed as essentially different extensive forms.

The questions raised above show the need for a rigorous approach to the question of how, and if, one can translate any given extensive form (a precise definition of extensive form will be given below, in sections 3 and 4) into a set of well-formed formulas of propositional calculus that are "true" for the extensive form. This is the object of this paper, which is organized as follows[5]. In section 2 we give a brief review of propositional calculus and of the so-called topological interpretation (or semantics), which is different from the standard one based on truth tables. In section 3 we describe how to obtain the logical description of an extensive form with perfect information and prove that such a description is "true" for the extensive form in terms of the topological semantics of section 2. We also establish – using the standard truth-table semantics – a relationship between the set of propositions that describe an extensive form and the set of plays in the extensive form. In section 4 we extend the analysis to general extensive games. The reason for a separate section on games with perfect information is that the symbolism needed is less complex and the analysis more straight-forward: having gone through the case of perfect-information games, it becomes easier to see through the more complex notation needed to deal with general extensive forms. Section 5 contains some concluding remarks.

2 The Topological Interpretation of Propositional Calculus

In this section we briefly review the definition of propositional calculus and the so-called topological interpretation, or semantics, of it[6].

Let S be a countable set (finite or infinite). The elements of S will be called *atomic sentences* and will be denoted by $A, B, \ldots$ (with or without subscript and/or superscript). A *propositional calculus based on* S consists of the following elements:

(1) An alphabet $\mathscr{A}$, which is the union of the set of atomic sentences S, the set of *connectives* $\{\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow\}$ and the set of *parentheses* $\{(\,,\,)\}$. A finite string of elements of $\mathscr{A}$ is called a *word*.

[5] Section 4 of Bonanno (1991b) sketches some of the ideas developed in this paper.
[6] See Rasiowa and Sikorski (1968). (I am grateful to the referees for pointing out this reference to me.)

(2) A set of "meaningful words", called *well-formed formulas* (wffs), which is the smallest set $\boldsymbol{W}$ of words satisfying the following properties:
   (i) If P is an atomic sentence then $(P) \in \boldsymbol{W}$,
   (ii) if $\Phi \in \boldsymbol{W}$ then $(\neg \Phi) \in \boldsymbol{W}$,
   (iii) if Φ, $\Psi \in \boldsymbol{W}$ then $(\Phi \vee \Psi) \in \boldsymbol{W}$, $(\Phi \wedge \Psi) \in \boldsymbol{W}$, $(\Phi \Rightarrow \Psi) \in \boldsymbol{W}$, and $(\Phi \Leftrightarrow \Psi) \in \boldsymbol{W}$.
(3) A set of wffs, called *axioms*, which is the union of the following sets (called *axiom schemes*)[7]:
   Axiom Scheme 1: $\{\Phi \vee \Phi \Rightarrow \Phi \mid \Phi \text{ is a wff}\}$.
   Axiom Scheme 2: $\{\Phi \Rightarrow \Phi \vee \Psi \mid \Phi \text{ and } \Psi \text{ are wffs}\}$.
   Axiom Scheme 3: $\{(\Phi \Rightarrow \Psi) \Rightarrow ((\Theta \vee \Phi) \Rightarrow (\Theta \vee \Psi)) \mid \Phi, \Psi \text{ and } \Theta \text{ are wffs}\}$.
(4) The rule of inference *Modus Ponens* which, for any wffs Φ and Ψ, allows one to infer Ψ from Φ and $\Phi \Rightarrow \Psi$.

Throughout the paper we shall use P (with or without subscript) as a placeholder for atomic sentences and Φ, Ψ, Θ, Δ (with or without subscript) as placeholders for wffs.

If Φ is a wff, a *proof of* Φ is a finite sequence $\Phi_1, \ldots, \Phi_m$ of wffs such that:

(1) $\Phi_m = \Phi$,
(2) for each $j = 1, \ldots, m$ either
   (i) Φ_j is an axiom, or
   (ii) there are $i<j$ and $k<j$ such that Φ_j is inferred by Modus Ponens from Φ_i and Φ_k.

A *theorem* is a wff which has a proof. We shall write $\vdash \Phi$ to denote that Φ is a theorem[8]. We say that Φ *logically implies* Ψ if $\vdash (\Phi \Rightarrow \Psi)$ and that two wffs Φ and Ψ are *logically equivalent* if $\vdash (\Phi \Leftrightarrow \Psi)$.

We now define the notion of topological semantics for propositional calculus (cf. Rasiowa and Sikorski, 1968).

Definition: An *topological interpretation* (for the propositional calculus based on the set S) is a pair $\langle \Omega, f \rangle$, where Ω is a non-empty set and $f: S \to \mathscr{P}(\Omega)$ is a function from the set of atomic sentences S into the set of subsets of Ω. Given a topological interpretation $\langle \Omega, f \rangle$, by induction on the construction of wffs (cf. Lightstone, 1978, p. 27) the following rules define a unique extension $\mathscr{F}: \boldsymbol{W} \to \mathscr{P}(\Omega)$ of f to the set of wffs:

(1) $\mathscr{F}[(P)] = f[P]$ if P is an atomic sentence;
(2) $\mathscr{F}[(\neg \Phi)] = \overline{\mathscr{F}[\Phi]}$ (where, for every set $V \subseteq \Omega$, $\overline{V}$ denotes the complement of V with respect to Ω, i.e. $\overline{V} = \Omega \setminus V$);

[7] A proof that axiom schemes 1–3 are sufficient for propositional calculus can be found in Lightstone (1978, pp. 37ff.).
[8] Note that '$\vdash$' is a symbol of the metalanguage, not a symbol of the object language.

(3) $\mathscr{F}[(\Phi \vee \Psi)] = \mathscr{F}[\Phi] \cup \mathscr{F}[\Psi]$;
(4) $\mathscr{F}[(\Phi \wedge \Psi)] = \mathscr{F}[\Phi] \cap \mathscr{F}[\Psi]$;
(5) $\mathscr{F}[(\Phi \Rightarrow \Psi)] = \overline{\mathscr{F}[\Phi]} \cup \mathscr{F}[\Psi]$;
(6) $\mathscr{F}[(\Phi \Leftrightarrow \Psi)] = (\overline{\mathscr{F}[\Phi]} \cap \overline{\mathscr{F}[\Psi]}) \cup (\mathscr{F}[\Phi] \cap \mathscr{F}[\Psi])$.

Definition: A topological interpretation $\langle \Omega, f \rangle$ is said to be a *model of* Φ, where Φ is a wff, if $\mathscr{F}[\Phi] = \Omega$. In this case we also say that Φ *is true in the interpretation* $\langle \Omega, f \rangle$.

The following theorems are well-known among logicians and will be stated without proof (proofs can be found in Rasiowa and Sikorski, 1968, and Bonanno, 1992b).

Soundness Theorem: If $\vdash \Phi$ (that is, if the wff Φ is a theorem of propositional calculus), then $\mathscr{F}[\Phi] = \Omega$ for every topological interpetation $\langle \Omega, f \rangle$.

Completeness Theorem: If Φ is a wff such that $\mathscr{F}[\Phi] = \Omega$ for every topological interpretation $\langle \Omega, f \rangle$, then $\vdash \Phi$.

3 The Logical Representation of an Extensive Form with Perfect Information

In this section we show how to associate with every extensive form with perfect information a set of wffs that are true for the extensive form, that is, for which the extensive form itself yields a topological model[9]. First we remind the reader of the definition of extensive form with perfect information.

A *directed graph* (or *digraph*) is a pair $(T, \rightarrow)$ where T is a *finite* set of *nodes* and $\rightarrow$ is an irreflexive binary relation on T. If $x \in T$, $y \in T$ and $x \rightarrow y$ we say that x is an *immediate predecessor* of y and y is an *immediate successor* of x; furthermore, we call the ordered pair xy an *arc* [as is customary in graph theory, we use the notation xy rather than (x, y)]. If $\eta = xy$ is an arc, we say that η is *incident from* x and *incident to* y. The *indegree* of a node t is the number of immediate predecessors of t. The *outdegree* of a node t is the number of immediate successors of t. A node is a *source* if it has positive outdegree and zero indegree. A node is a *terminal node* if it has positive indegree and zero outdegree. Let $\eta_1 = x_1 y_1$ and $\eta_2 = x_2 y_2$ be two arcs. We say that η_1 is *adjacent to* η_2 if $y_1 = x_2$. Let $\langle \eta_1 = x_1 y_1, \ldots, \eta_m = x_m y_m \rangle$ be a finite sequence of arcs ($m \geq 2$). If, for every $k = 1, \ldots, m-1$, η_k is adjacent to η_{k+1}, then we call the sequence a *path from* x_1 *to* y_m. A digraph $(T, \rightarrow)$ is a *tree from a point* if it has exactly one source and every other node has indegree one. It can be shown that in a tree from a point there is a unique path from the source to any other node.

[9] We shall assume throughout the paper that the extensive form has no chance moves. However, Nature can be treated just like any other player. The only difference is that one needs to add atomic sentences that describe the probabilities attached to Nature's choices.

Let $(T, \rightarrow)$ be a tree from a point. We shall denote by $Z \subset T$ the set of terminal nodes (it can be shown that $Z \neq \emptyset$) and we shall call the nodes in $T \backslash Z$ *decision nodes*. By a *play* we shall mean a path from the source to a terminal node.

An *extensive form with perfect information* is a triple $G = \langle (T, \rightarrow), N, \iota \rangle$ where:

(i) $(T, \rightarrow)$ is a tree from a point,
(ii) N is a finite set of *players* (Roman numerals will be used for, and only for, players), and
(iii) $\iota : T \backslash Z \rightarrow N$ is an onto function that associates a player with every decision node[10].

Given an extensive form with perfect information G we construct the associated *set of atomic sentences* $S(G)$ as follows. Label the nodes of the tree as $x_0, x_1, x_2, \ldots, x_m$ in such a way that if $x_j \rightarrow x_k$ then $j < k$ (thus x_0 is the source)[11]. With every arc $x_j x_k$ we associate an atomic sentence denoted by $A_{\iota(x_j),k}$. The set $S(G)$ consists of all and only such symbols. The intended interpretation of $A_{\iota(x_j),k}$ is "player $\iota(x_j)$ [the player who moves at node x_j] takes the action that leads from the immediate predecessor of node x_k to node x_k". For example, in the extensive form of Figure 3 each element of $S(G)$ is written next to the corresponding arc.

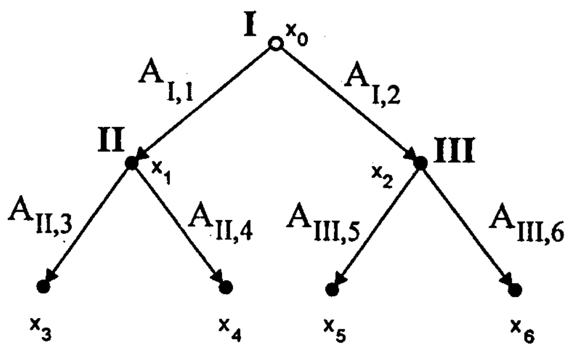

Fig. 3.

Given an extensive form with perfect information and the associated set of atomic sentences $S(G)$, consider the propositional calculus based on $S(G)$. We now associate with every decision node a wff of this propositional calculus as follows [from now on in every wff we shall omit the outermost brackets, so that, for example, we shall write $A \vee B$ rather than $(A \vee B)$]:

(1) Let $x_0 x_{k_1}, x_0 x_{k_2}, \ldots, x_0 x_{k_m}$ be the arcs incident from x_0 (the source) and $p = \iota(x_0)$ the player assigned to the source. Then the following wffs are associated with the source:

$A_{p,k_1} \vee A_{p,k_2} \vee \ldots \vee A_{p,k_m}$
$\neg(A_{p,k_i} \wedge A_{p,k_j})$ for all $i, j = 1, 2, \ldots, m$, with $i \neq j$.

[10] If we add to an extensive form with perfect information a *payoff function* $\pi_i : Z \rightarrow \Re$, for every player $i \in N$ (where $\Re$ denotes the set of real numbers), we obtain an *extensive game with perfect information*. In this paper we shall be concerned with extensive forms rather than extensive games.

[11] Such a labeling is possible: see Harary et al. (1965, Corollary 10.1.a, p. 269).

(2) If x_k is a decision node different from the source (that is, $k \neq 0$), $p = \imath(x_k)$ is the player assigned to node x_k, x_j is the immediate predecessor of x_k, $q = \imath(x_j)$ is the player assigned to node x_j, and $x_k x_{r_1}, x_k x_{r_2}, \ldots, x_k x_{r_n}$ are the arcs incident from x_k then the following wffs are associated with node x_k:

$$A_{q,k} \Leftrightarrow (A_{p,r_1} \vee A_{p,r_2} \vee \ldots \vee A_{p,r_n})$$
$$\neg(A_{p,r_s} \wedge A_{p,r_t}) \text{ for all } s, t = 1, 2, \ldots, n \text{ with } s \neq t.$$

Example: For the extensive form of Figure 3, we have:

(1) wffs associated with the source: $A_{\mathrm{I},1} \vee A_{\mathrm{I},2}$, $\neg(A_{\mathrm{I},1} \wedge A_{\mathrm{I},2})$
(2) wffs associated with node x_1: $A_{\mathrm{I},1} \Leftrightarrow (A_{\mathrm{II},3} \vee A_{\mathrm{II},4})$, $\neg(A_{\mathrm{II},3} \wedge A_{\mathrm{II},4})$
(3) wffs associated with node x_2: $A_{\mathrm{I},2} \Leftrightarrow (A_{\mathrm{III},5} \vee A_{\mathrm{III},6})$, $\neg(A_{\mathrm{III},5} \wedge A_{\mathrm{III},6})$.

Let $\Gamma(G)$ be the set of wffs associated with the decision nodes of G. We now want to show that there is a precise sense in which the wffs of $\Gamma(G)$ are true for G.

Let E be the set of arcs. Above we constructed a one-to-one map from E onto the set of atomic sentences $S(G)$. Let h be the inverse of this map, that is, $h: S(G) \rightarrow E$ is the map that associates with every atomic sentence in $S(G)$ the arc from which it was obtained. Finally, let $\lambda: E \rightarrow \mathscr{P}(Z)$ be the map that associates with every arc the set of *terminal* nodes that are reached by plays that contain that arc[12].

Proposition 3.1: Let G an extensive form with perfect information. Let $\Gamma(G)$ be the set of wffs associated with the decision nodes of G. Then the topological interpretation $\langle \Omega, f \rangle$ where $\Omega = Z$ and $f = \lambda \circ h$ is a model for $\Gamma(G)$, that is, it is a model for every wff in $\Gamma(G)$.

For a rigorous proof we refer the reader to the Appendix. Here we shall give an illustration based on the extensive form of Figure 3, for which $\Gamma(G) = \{A_{\mathrm{I},1} \vee A_{\mathrm{I},2}, \neg(A_{\mathrm{I},1} \wedge A_{\mathrm{I},2}), A_{\mathrm{I},1} \Leftrightarrow (A_{\mathrm{II},3} \vee A_{\mathrm{II},4}), \neg(A_{\mathrm{II},3} \wedge A_{\mathrm{II},4}), A_{\mathrm{I},2} \Leftrightarrow (A_{\mathrm{III},5} \vee A_{\mathrm{III},6}), \neg(A_{\mathrm{III},5} \wedge A_{\mathrm{III},6})\}$. We want to show that for every $\Phi \in \Gamma(G)$, $(\lambda \circ h)(\Phi) = Z$, where $Z = \{x_3, x_4, x_5, x_6\}$.

(i) $(\lambda \circ h)(A_{\mathrm{I},1} \vee A_{\mathrm{I},2}) = (\lambda \circ h)(A_{\mathrm{I},1}) \cup (\lambda \circ h)(A_{\mathrm{I},2}) = \lambda(x_0 x_1) \cup \lambda(x_0 x_2) = \{x_3, x_4\} \cup \{x_5, x_6\} = Z$.
(ii) $(\lambda \circ h)(\neg(A_{\mathrm{I},1} \wedge A_{\mathrm{I},2}) = \overline{(\lambda \circ h)(A_{\mathrm{I},1} \wedge A_{\mathrm{I},2})} = \overline{(\lambda \circ h)(A_{\mathrm{I},1}) \cap (\lambda \circ h)(A_{\mathrm{I},2})} = \overline{\lambda(x_0 x_1) \cap \lambda(x_0 x_2)} = \overline{\{x_3, x_4\} \cap \{x_5, x_6\}} = \overline{\varnothing} = Z$.

[12] For example, in the extensive form of Figure 3, $\lambda(x_0 x_1) = \{x_3, x_4\}$, $\lambda(x_0 x_2) = \{x_5, x_6\}$, $\lambda(x_1 x_3) = \{x_3\}$, $\lambda(x_1 x_4) = \{x_4\}$, $\lambda(x_2 x_5) = \{x_5\}$, $\lambda(x_2 x_6) = \{x_6\}$.

(iii) $(\lambda \circ h)(A_{I,1} \Leftrightarrow (A_{II,3} \vee A_{II,4})) = ((\overline{(\lambda \circ h)(A_{I,1})} \cap \overline{(\lambda \circ h)(A_{II,3} \vee A_{II,4})}) \cup$
$((\lambda \circ h)(A_{I,1})$
$\cap (\lambda \circ h)(A_{II,3} \vee A_{II,4})) = (\overline{\lambda(x_0 x_1)} \cap \overline{\lambda(x_1 x_3) \cup \lambda(x_1 x_4)}) \cup (\lambda(x_0 x_1)$
$\cap (\lambda(x_1 x_3) \cup \lambda(x_1 x_4))) = (\{x_5, x_6\} \cap \{x_5, x_6\}) \cup (\{x_3, x_4\} \cap \{x_3, x_4\}) = Z.$

(iv) $(\lambda \circ h)(\neg(A_{II,3} \wedge A_{II,4})) = \overline{(\lambda \circ h)(A_{II,3} \wedge A_{II,4})} = \overline{(\lambda \circ h)(A_{II,3}) \cap (\lambda \circ h)(A_{II,4})}$
$= \overline{\lambda(x_1 x_3) \cap \lambda(x_1 x_4)} = \overline{\{x_3\} \cap \{x_4\}} = \overline{\varnothing} = Z.$

(v) $(\lambda \circ h)(A_{I,2} \Leftrightarrow (A_{III,5} \vee A_{III,6})) = ((\overline{(\lambda \circ h)(A_{I,2})} \cap \overline{(\lambda \circ h)(A_{III,5} \vee A_{II,6})})$
$\cup ((\lambda \circ h)(A_{I,2}) \cap (\lambda \circ h)(A_{III,5} \vee A_{III,6})) = (\overline{\lambda(x_0 x_2)} \cap \overline{\lambda(x_2 x_5) \cup \lambda(x_2 x_6)})$
$\cup (\lambda(x_0 x_2) \cap (\lambda(x_2 x_5) \cup \lambda(x_2 x_6))) = (\{x_3, x_4\} \cap \{x_3, x_4\})$
$\cup (\{x_5, x_6\} \cap \{x_5, x_6\}) = Z.$

(vi) $(\lambda \circ h)(\neg(A_{III,5} \wedge A_{III,6})) = \overline{(\lambda \circ h)(A_{III,5} \wedge A_{III,6})}$
$= \overline{(\lambda \circ h)(A_{III,5}) \cap (\lambda \circ h)(A_{III,6})}$
$= \overline{\lambda(x_2 x_5) \cap \lambda(x_2 x_6)} = \overline{\{x_5\} \cap \{x_6\}} = \overline{\varnothing} = Z$

The next proposition shows that there is another sense in which the wffs in $\Gamma(G)$ are true for G: the valuations that satisfy *all* the wffs in $\Gamma(G)$ are in one-to-one correspondence with the plays of G[13]. First we recall the definition of valuation. Given a propositional calculus based on the set S of atomic sentences, a *valuation* is a function $v: S \to \{T, F\}$ (where T strands for "true" and F for "false"). By induction on the construction of wffs the following rules define a unique extension $V: \boldsymbol{W} \to \{T, F\}$ of v to the set of wffs[14]:

(a) $V[(P)] = v[P]$ for every atomic sentence P;

(b) $V[(\neg \Psi)] = \begin{cases} T \text{ if } V[\Psi] = F \\ F \text{ if } V[\Psi] = T \end{cases}$

(c) $V[(\Psi \vee \Theta)] = \begin{cases} T \text{ if } V[\Psi] = T \text{ or } V[\Theta] = T \\ F \text{ if } V[\Psi] = V[\Theta] = F \end{cases}$

Proposition 3.2: Let G be an extensive form with perfect information and $S(G)$ be the associated set of atomic sentences. Let $\Gamma(G)$ be the set of wffs associated with the decision nodes of G. Let $\mathscr{V}$ be the set of valuations that satisfy all the wffs of $\Gamma(G)$, this is,

$$\mathscr{V} = \{v: S(G) \to \{T, F\} \mid V(\Phi) = T \text{ for all } \Phi \in \Gamma(G)\}.$$

Then there is a one-to-one and onto map between the set $\mathscr{V}$ and the set of plays of G.

Again, a rigorous proof is given in the Appendix. Here we give an illustration based on Figure 3. Let Φ be the conjunction of the wffs in $\Gamma(G)$, that is, $\Phi = (A_{I,1} \vee A_{I,2}) \wedge (\neg(A_{I,1} \wedge A_{I,2})) \wedge (A_{I,1} \Leftrightarrow (A_{II,3} \vee A_{II,4})) \wedge (\neg(A_{II,3} \wedge A_{II,4})) \wedge (A_{I,2} \Leftrightarrow (A_{III,5} \vee A_{III,6})) \wedge (\neg(A_{III,5} \wedge A_{III,6}))$. Each row in the following table corresponds to a valuation v such that $V(\Phi) = T$ (it is easy to check that these are the only valuations

[13] Cf. the remark in Bonanno (1991a), page 42.
[14] Using (a)–(c) one then shows that $V[(\Psi \wedge \Theta)] = T$ if and only if $V[\Psi] = V[\Theta] = T$, etc.

that satisfy this property). The last column in the table shows the corresponding play [for sample, when $A_{I,1}$ and $A_{II,4}$ are given thruth value T (second row), we pick the corresponding arcs, namely x_0x_1 and x_1x_4, respectively, and we obtain the play from x_0 to x_4]:

Table 1.

$A_{I,1}$	$A_{I,2}$	$A_{II,3}$	$A_{II,4}$	$A_{III,5}$	$A_{III,6}$	Φ	PLAY from x_0 to
T	F	T	F	F	F	T	x_3
T	F	F	T	F	F	T	x_4
F	T	F	F	T	F	T	x_5
F	T	F	F	F	T	T	x_6

We can now use the analysis of this section to discuss the examples of Figures 1 and 2. Concerning the extensive form of Figure 1, it is easy to check that – according to the topological interpretation suggested in this paper – propositon (3) rather than proposition (3b) of section 1 is true for the extensive form[15]. Thus propositions (1)–(4) of section 1 give indeed a correct description of the extensive form of Figure 1 (to be entirely rigorous, one would have to change the notation used in (1)–(4) to match the symbolism introduced in this section). Turning now to Figure 2, the two extensive forms G_1 and G_2 have the same tree and differ only in the function that assigns players to decision nodes. Let us choose a labeling of the nodes of the common tree, as shown in Figure 4. We can then construct the sets of atomic sentences as described above and obtain the sets $S(G_1)$ and $S(G_2)$ respectively, where the elements of $S(G_1)$ are written next to the arcs of G_1 in Figure 4 and similarly for $S(G_2)$. It is clear that the two sets $S(G_1)$ and $S(G_2)$ are different: they are even disjoint. In other words, the alphabet used to describe extensive form G_1 is different from the alphabet used to describe extensive form G_2. It follows that the description of the two extensive forms are not logically equivalent. It could be objected that it is not fair to compare Figures 2 and 4: for example, in Figure 4 the label '$A_{I,1}$' occurs only in G_1, while its counterpart in Figure 2 – the label 'a' – appears both in G_1 and G_2. However, it should be noted that a labeling of the arcs, although useful, *is not part of the definition of extenesive form.* Thus whenever one adds such a labeling, one introduces an element of arbitrariness. We have put forward a rule that can be used to label unambiguously the arcs of any extensive form (with perfect information). *Using this rule,* games G_1 and G_2 are labeled differently and therefore do not have the same logical representation.

[15] This does not imply that there cannot be a *different* interpretation (or semantics), on the basis of which (3b) would be true while (3) would not.

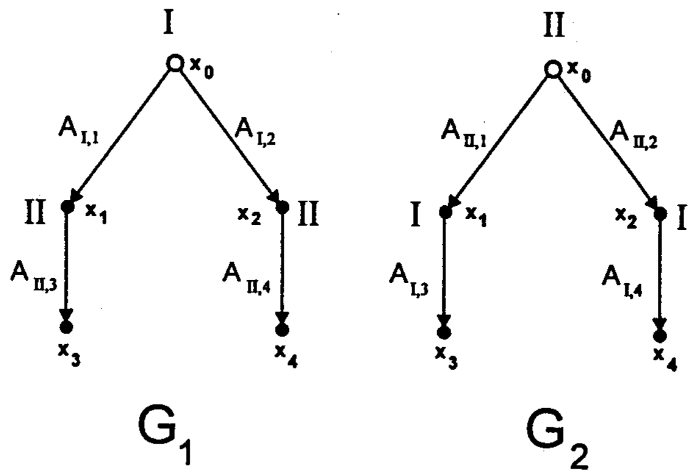

Fig. 4.

4 The Logical Description of a General Extensive Form

We now consider general extensive forms, that is, extensive forms with perfect or imperfect information.

An *extensive form* is a tuple $G=\langle(T, \rightarrow), N, \imath, (\mathscr{H}_i)_{i\in N}, \mathscr{C}\rangle$ where:

(i) $(T, \rightarrow)$ is a tree from a point,
(ii) N is a finite set of *players*,
(iii) $\imath: T\backslash Z \rightarrow N$ is an onto function that associates a player with every decision node,
(iv) for every player $i\in N$, $\mathscr{H}_i$ is a partition of the decision nodes of player i into *information sets of player i* satisfying the following properties:
  (a) if $h\in\mathscr{H}_i$, $x\in h$ and $y\in h$, then the outdegree of x is equal to the outdegree of y, and
  (b) if $h\in\mathscr{H}_i$, $x\in h$ and $y\in h$, then there is no path from x to y or from y to x,
(v) $\mathscr{C}$ is a partition of the arcs of $(T, \rightarrow)$ into *choices* satisfying the following properties. If $xy\in c\in\mathscr{C}$ and x belongs to information set h, then:
  (a) every arc in c is incident from a node in h, and
  (b) for every node w in h, there is one and only one arc indicent from w that belongs to c[16].

An extensive form has *perfect information* if every information set (and therefore every choice) is a singleton. In such a case the definition can be simplified to the one given in section 3. An extensive form has *imperfect information* if at least one information set contains at least two nodes.

[16] As we remarked before, it we add to an extensive form a *payoff function* $\pi_i: Z\rightarrow\Re$, for every player $i\in N$, we obtain an *extensive game*.

Given an extensive form G we construct the associated *set of atomic sentences* $S(G)$ as follows. As before, label the nodes of the tree as $x_0, x_1, x_2, \ldots, x_m$ in such a way that if $x_j \rightarrow x_k$ then $j<k$ (thus x_0 is the source). Fix an arbitrary choice $c=\{x_{j_1}x_{k_1}, x_{j_2}x_{k_2}, \ldots, x_{j_n}x_{k_n}\}$. Then $\{x_{j_1}, x_{j_2}, \ldots, x_{j_n}\}$ is an information set of a player, say player i. Let $k=\min\{k_1, k_2, \ldots, k_n\}$. Associate with every arc in c the atomic sentence $A_{i,k}$. The set $S(G)$ consists of all and only such symbols.

Example: Consider the extensive form of Figure 5. In this case there are four information sets (represented by rectangles): $\{x_0\}$, $\{x_1\}$, $\{x_2, x_3\}$ and $\{x_4, x_5\}$. Let the choices be as follows: $\{x_0x_1\}$, $\{x_0x_2\}$, $\{x_0x_3\}$, $\{x_1x_6\}$, $\{x_1x_4\}$, $\{x_2x_5, x_3x_{12}\}$, $\{x_2x_{11}, x_3x_{13}\}$, $\{x_4x_7, x_5x_9\}$, $\{x_4x_8, x_5x_{10}\}$. The atomic sentences for this game are shown in Figure 5 next to the corresponding arcs.

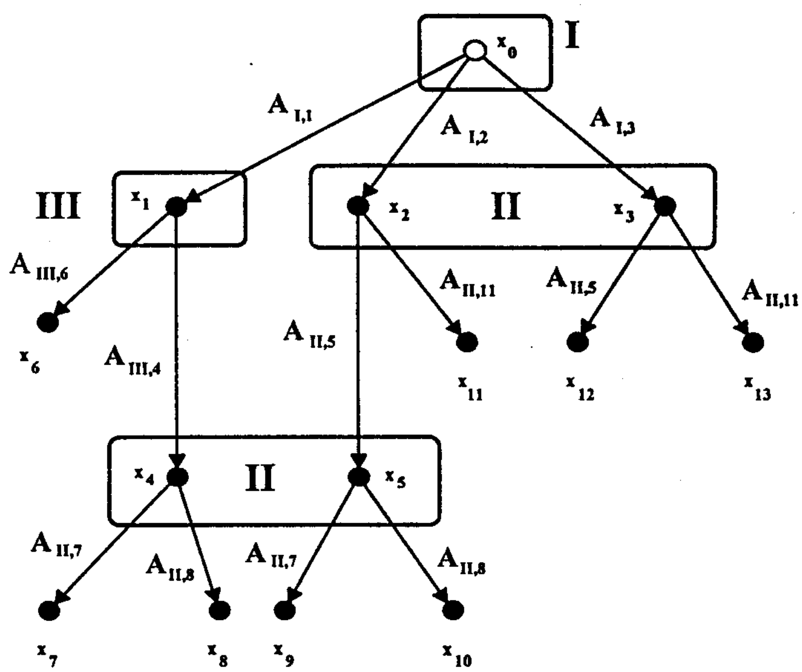

Fig. 5.

Given an extensive form and the associated set of atomic sentences, consider the propositional calculus based on $S(G)$. We now associate with every information set a wff of this propositional calculus as follows:

(1) Let $x_0x_{k_1}, x_0x_{k_2}, \ldots, x_0x_{k_m}$ be the arcs incident from x_0 (the source) and $p=\iota(x_0)$ the player assigned to the source. Then the following wffs are associated with the source:

$A_{p,k_1} \vee A_{p,k_2} \vee \ldots \vee A_{p,k_m}$

$\neg(A_{p,k_i} \wedge A_{p,k_j})$ for all $i, j=1, 2, \ldots, m$, with $i \neq j$

(2) Let $h=\{y_1, y_2, \ldots, y_m\}$ be an information set such that $x_0 \notin \{y_1, y_2, \ldots, y_m\}$, and p be the corresponding player. Let there be r choices at h. Let $A_{p,k_1}, A_{p,k_2}, \ldots, A_{p,k_r}$ be the atomic sentences associated with the choices at h. Then the following wffs arc associated with information set h:

$\neg(A_{p,k_i} \wedge A_{p,k_j})$ for all $i, j=1, 2, \ldots, r$, with $i \neq j$.

Furthermore, for every node $y_j \in h$, let $B^{j,1}, \ldots, B^{j,n_j}$ be the atomic sentences associated with the arcs that form the path from the source to y_j. Then the following wff is associated with h:

$$(A_{p,k_1} \vee A_{p,k_2} \vee \ldots \vee A_{p,k_r}) \Leftrightarrow (B^{1,1} \wedge \ldots \wedge B^{1,n_1}) \vee (B^{2,1} \wedge \ldots \wedge B^{2,n_2}) \vee \ldots \vee (B^{m,1} \wedge \ldots \wedge B^{m,n_m})$$

Example: For the extensive form of Figure 5, we have:

(1) wffs associated with the source: $A_{I,1} \vee A_{I,2} \vee A_{I,3}$, $\neg(A_{I,1} \wedge A_{I,2})$, $\neg(A_{I,1} \wedge A_{I,3})$, $\neg(A_{I,2} \wedge A_{I,3})$;
(2) wffs associated with $\{x_1\}$: $\neg(A_{III,4} \wedge A_{III,6})$, $A_{I,1} \Leftrightarrow (A_{III,4} \vee A_{III,6})$;
(3) wffs associated with $\{x_2, x_3\}$: $\neg(A_{II,5} \wedge A_{II,11})$, $(A_{II,5} \vee A_{II,11}) \Leftrightarrow (A_{I,2} \vee A_{I,3})$;
(4) wffs associated with $\{x_4, x_5\}$: $\neg(A_{II,7} \wedge A_{II,8})$, $(A_{II,7} \vee A_{II,8}) \Leftrightarrow ((A_{I,1} \wedge A_{III,4}) \vee (A_{I,2} \wedge A_{II,5}))$.

Let $\Gamma(G)$ be the set of wffs associated with the information sets of G. We now show that there is a precise sense in which the wffs of $\Gamma(G)$ are true for G. Let $\mathscr{C}$ be the set of choices. Above we constructed a one-to-one map from $\mathscr{C}$ onto the set of atomic sentences $S(G)$. Let $\tilde{h}$ be the inverse of this map, that is, $\tilde{h}: S(G) \to \mathscr{C}$ is the map that associates with every atomic sentence in $S(G)$ the choice from which it was obtained. Finally, let $\tilde{\lambda}$: $\mathscr{C} \to \mathscr{P}(Z)$ be the map that associates with every choice the set of terminal nodes that are reached by plays that contain an arc that belongs to that choice[17].

Proposition 4.1: Fix an extensive form G. Let $\Gamma(G)$ be the set of wffs associated with the information sets of G. Then the topological interpretation $\langle \Omega, f \rangle$ where $\Omega = Z$ and $f = \tilde{\lambda} \circ \tilde{h}$ is a model for $\Gamma(G)$, that is, it is a model for every wff in $\Gamma(G)$.

As before, we have relegated the rigorous proof to the Appendix. Here we shall give an illustration based on Figure 5. However, in order to economize on space, we shall show that $(\tilde{\lambda} \circ \tilde{h})(\Phi) = Z = \{x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}$ only for the wffs Φ associated with information set $\{x_2, x_3\}$.

(i) $(\tilde{\lambda} \circ \tilde{h})(\neg(A_{II,5} \wedge A_{II,11})) = \overline{(\tilde{\lambda} \circ \tilde{h})(A_{II,5} \wedge A_{II,11})} = \overline{(\tilde{\lambda} \circ \tilde{h})(A_{II,5}) \cap (\tilde{\lambda} \circ \tilde{h})(A_{II,11})}$
$= \overline{\tilde{\lambda}(\{x_2x_5, x_3x_{12}\}) \cap \tilde{\lambda}(\{x_2x_{11}, x_3x_{13}\})} = \overline{\{x_9, x_{10}, x_{12}\} \cap \{x_{11}, x_{13}\}} = \overline{\varnothing} = Z.$

(ii) $(\tilde{\lambda} \circ \tilde{h})((A_{II,5} \vee A_{II,11}) \Leftrightarrow (A_{I,2} \vee A_{I,3})) = (\overline{(\tilde{\lambda} \circ \tilde{h})(A_{II,5} \vee A_{II,11})}$
$\cap \overline{(\tilde{\lambda} \circ \tilde{h})(A_{I,2} \vee A_{I,3})}) \cup ((\tilde{\lambda} \circ \tilde{h})(A_{II,5} \vee A_{II,11}) \cap (\tilde{\lambda} \circ \tilde{h})(A_{I,2} \vee A_{I,3}))$
$= (\overline{\tilde{\lambda}(\{x_2x_5, x_3x_{12}\}) \cup \tilde{\lambda}(\{x_2x_{11}, x_3x_{13}\})} \cap \overline{\tilde{\lambda}(\{x_0x_2\}) \cup \tilde{\lambda}(\{x_0x_3\})})$
$\cup ((\tilde{\lambda}(\{x_2x_5, x_3x_{12}\}) \cup \tilde{\lambda}(\{x_2x_{11}, x_3x_{13}\})) \cap ((\tilde{\lambda}(\{x_0x_2\}) \cup \tilde{\lambda}(\{x_0, x_3\})))$
$= (\overline{\{x_9, x_{10}, x_{12}\} \cup \{x_{11}, x_{13}\}} \cap \overline{\{x_9, x_{10}, x_{11}\} \cup \{x_{12}, x_{13}\}}) \cup ((\{x_9, x_{10}, x_{12}\}$
$\cup \{x_{11}, x_{13}\}) \cap (\{x_9, x_{10}, x_{11}\} \cup \{x_{12}, x_{13}\})) = \overline{\{x_9, x_{10}, x_{11}, x_{12}, x_{13}\}}$
$\cup \{x_9, x_{10}, x_{11}, x_{12}, x_{13}\} = Z.$

---

[17] For example, in the extensive form of Figure 5, $\tilde{\lambda}(\{x_0x_1\}) = \{x_6, x_7, x_8\}$, $\tilde{\lambda}(\{x_2x_5, x_3x_{12}\}) = \{x_9, x_{10}, x_{12}\}$, $\tilde{\lambda}(\{x_2x_{11}, x_3x_{13}\}) = \{x_{11}, x_{13}\}$, etc.

Proposition 4.2: Let G be an extensive form and S(G) be the associated set of atomic sentences. Let $\Gamma(G)$ be the set of wffs associated with the information sets of G. Let $\mathscr{V}$ be the set of valuations that satisfy all the wffs of $\Gamma(G)$, that is,

$$\mathscr{V} = \{v : S(G) \rightarrow \{T, F\} \mid V(\Phi) = T \text{ for all } \Phi \in \Gamma(G)\}.$$

Then there is a one-to-one and onto map between the set $\mathscr{V}$ and the set of plays of G.

Proof: Again, for a rigorous proof the reader is referred to the Appendix. Here we shall give an illustration based on Figure 5. Let Φ be the conjunction of the wffs in $\Gamma(G)$, where $\Gamma(G)$ is the set of wffs given in (1)–(4) of the example preceding proposition 4.1. Each row in the following table corresponds to a valuation v such that $V(\Phi) = T$ (it is easy to check that these are the only valuations that satify this property). For greater clarity we have only written the truth-value T (thus every empty cell corresponds to truth-value F). The last column in the table shows the corresponding play [for example, when $A_{\mathrm{I},1}$, $A_{\mathrm{III},4}$ and $A_{\mathrm{II},8}$ are given truth value T (third row), we pick the corresponding arcs, namely x_0x_1, x_1x_4 and x_4x_8, respectively, and obtain the play from x_0 to x_8]:

Table 2.

$A_{\mathrm{I},1}$	$A_{\mathrm{I},2}$	$A_{\mathrm{I},3}$	$A_{\mathrm{II},5}$	$A_{\mathrm{II},11}$	$A_{\mathrm{III},6}$	$A_{\mathrm{III},4}$	$A_{\mathrm{II},7}$	$A_{\mathrm{II},8}$	Φ	PLAY from x_0 to
T					T				T	x_6
T						T	T		T	x_7
T						T		T	T	x_8
	T		T				T		T	x_9
	T		T					T	T	x_{10}
	T			T					T	x_{11}
		T	T						T	x_{12}
		T		T					T	x_{13}

5 Concluding Remarks

We showed how to associate with every extensive form G a set $\Gamma(G)$ of well-formed formulas of propositional calculus that are true for the extensive form. In other words, the extensive form yields a topological model of $\Gamma(G)$.

The step from extensive forms to extensive games is a simple one. An extensive game is obtained from an extensive form by adding a payoff function $\pi_i : Z \rightarrow \mathfrak{R}$ for each player $i \in N$ (recall that Z is the set of terminal nodes and $\mathfrak{R}$ is the set of real numbers). Given an extensive game of *perfect information* with n players, we extend the set of atomic sentences $S(G)$ by associating with each terminal node $z \in Z$ the n

sentences '$\pi_i = \pi_i(z)$' ($i \in N$) whose intended interpretation is "player i's payoff is the number $\pi_i(z)$". Finally, we extend $\Gamma(G)$ by adding for every arc $x_j x_k$ incident to a terminal node (thus $x_k \in Z$) the following n wffs: $(A_{\iota(x_j),k} \Rightarrow \pi_i = \pi_i(x_k))$ ($i \in N$). Similarly, given a general extensive game, for every terminal node $z \in Z$ we would add the following n wffs: $(B^1 \wedge B^2 \wedge \ldots \wedge B^r \Rightarrow \pi_i = \pi_i(x_k))$, where $B^1, B^2, \ldots, B^r$ are the atomic sentences associated with the arcs that form the play from the source to z.

Appendix

In this appendix we prove propositions 3.1, 3.2, 4.1 and 4.2.

Proof of Proposition 3.1: First we establish the following facts.

Fact 1: Since in a tree from a point there is a path from the source to any other node, the set of terminal nodes can be reached from the source. Thus if $x_0 x_{k_1}$, $x_0 x_{k_2}, \ldots, x_0 x_{k_m}$ are the arcs incident from x_0,

$$\lambda(x_0 x_{k_1}) \cup \lambda(x_0 x_{k_2}) \cup \ldots \cup \lambda(x_0 x_{k_m}) = Z \tag{A.1}$$

Fact 2: Since in a tree from a point for every node $x \neq x_0$ there is a unique path from x_0 to x, if $x_j x_r$ and $x_j x_s$ are two different arcs incident from node x_j (thus $r \neq s$),

$$\lambda(x_j x_r) \cap \lambda(x_j x_s) = \varnothing \tag{A.2}$$

Fact 3: By definition of path, the set of terminal nodes that can be reached starting from node x_j following arc $x_j x_k$, is equal to the set of terminal nodes that can be reached from node x_k. Thus if $x_k x_{r_1}, x_k x_{r_2}, \ldots, x_k x_{r_n}$ are the arcs incident from x_k

$$\lambda(x_j x_k) = \lambda(x_k x_{r_1}) \cup \lambda(x_k x_{r_2}) \cup \ldots \cup \lambda(x_k x_{r_n}) \tag{A.3}$$

Now, proposition 3.1 is an immediate consequence of (A.1)–(A.3). In fact, let $x_0 x_{k_1}, x_0 x_{k_2}, \ldots, x_0 x_{k_m}$ be the arcs incident from the source and let

$$A_{p,k_1} \vee A_{p,k_2} \vee \ldots \vee A_{p,k_m}$$
$$\neg(A_{p,k_i} \wedge A_{p,k_j}) \text{ for all } i, j = 1, 2, \ldots, m, \text{ with } i \neq j$$

be the wffs associated with the source. Then

$$\begin{aligned}
&(\lambda \circ h)(A_{p,k_1} \vee A_{p,k_2} \vee \ldots A_{p,k_m}) \\
&= ((\lambda \circ h)(A_{p,k_1})) \cup ((\lambda \circ h)(A_{p,k_2})) \cup \ldots \cup ((\lambda \circ h)(A_{p,k_m})) \\
&= \lambda(x_0 x_{k_1}) \cup \lambda(x_0 x_{k_2}) \cup \ldots \cup \lambda(x_0 x_{k_m}) = Z \text{ [by (A.1)].}
\end{aligned}$$

Similarly, for every $i, j = 1, \ldots, m$ with $i \neq j$,

$$\begin{aligned}&(\lambda \circ h)(\neg(A_{p,k_i} \cap A_{p,k_j})) = \overline{(\lambda \circ h)(A_{p,k_i} \wedge A_{p,k_j})}\\ &= \overline{(\lambda \circ h)(A_{p,k_i}) \cap (\lambda \circ h)(A_{p,k_j})} = \overline{\lambda(x_0 x_{k_i}) \cap \lambda(x_0 x_{k_j})} = \overline{\varnothing} = Z \text{ [by (A.2)].}\end{aligned}$$

Let $x_k \neq x_0$ be a decision node, x_j be the immediate predecessor of x_k and $x_k x_{r_1}$, $x_k x_{r_2}, \ldots, x_k x_{r_n}$ be the arcs incident from x_k and let

$$\begin{aligned}&A_{q,k} \Leftrightarrow (A_{p,r_1} \vee A_{p,r_2} \vee \ldots \vee A_{p,r_n})\\ &\neg(A_{p,r_s} \wedge A_{p,r_t}) \text{ for all } s, t = 1, 2, \ldots, n, \text{ with } s \neq t\end{aligned}$$

be the wffs associated with node x_k. Then

$$\begin{aligned}&(\lambda \circ h)(A_{q,k} \Leftrightarrow (A_{p,r_1} \vee A_{p,r_2} \vee \ldots \vee A_{p,r_n}))\\ &= ((\overline{(\lambda \circ h)(A_{q,k})} \cap \overline{(\lambda \circ h)(A_{p,r_1} \vee A_{p,r_2} \vee \ldots \vee A_{p,r_n})})\\ &\quad \cup ((\lambda \circ h)(A_{q,k}) \cap (\lambda \circ h)(A_{p,r_1} \vee \ldots \vee A_{p,r_n}))\\ &= (\overline{\lambda(x_j x_k)} \cap \overline{(\lambda(x_k x_{r_1}) \cup \lambda(x_k x_{r_2}) \cup \ldots \cup \lambda(x_k x_{r_n})})\\ &\cup (\lambda(x_j x_k) \cap (\lambda(x_k x_{r_1}) \cup \lambda(x_k x_{r_2}) \cup \ldots \cup \lambda(x_k x_{r_n}))) = \text{[by (A.3)]}\\ &= \overline{\lambda(x_j x_k)} \cup \lambda(x_j x_k) = Z.\end{aligned}$$

□

Proof of Proposition 3.2: Fix an extensive form with perfect information G. First we show that if v is a valuation that satisfies all the wffs in $\Gamma(G)$ then there is a unique play in G associated with it. Let $E(v)$ be the set of arcs of G whose corresponding atomic sentences are assigned truth value T by v. First of all, there must be in $E(v)$ an arc incident from the source, otherwise the following wff of $\Gamma(G)$ would have truth value F (where $x_0 x_{k_1}, x_0 x_{k_2}, \ldots, x_0 x_{k_m}$ are the arcs incident from the source):

$$A_{\iota(x_0),k_1} \vee A_{\iota(x_0),k_2} \vee \ldots \vee A_{\iota(x_0),k_m}.$$

Furthermore, if $x_j x_k \in E(v)$, x_k is a decision node and $x_k x_{r_1}, x_k x_{r_2}, \ldots, x_k x_{r_n}$ are the arcs incident from x_k, then there must be an $s \in \{1, 2, \ldots, n\}$ such that $x_k x_{r_s} \in E(v)$, otherwise the following wff of $\Gamma(G)$ would have truth value F:

$$A_{\iota(x_j),k} \Leftrightarrow (A_{\iota(x_k),r_1} \vee A_{\iota(x_k),r_2} \vee \ldots \vee A_{\iota(x_k),r_n}).$$

Finally, if $x_j x_k \in E(v)$ and $x_r x_s \in E(v)$, with $k \neq s$, then $j \neq r$, otherwise the wff $\neg(A_{\iota(x_j),k} \wedge A_{\iota(x_j),s})$ would have truth value F.

Now we show the converse, namely that with every play in G we can associate a unique valuation v that satisfies all the wffs in $\Gamma(G)$. Fix an arbitrary play $\langle x_0 x_1, x_1 x_2, \ldots, x_{m-1} x_m \rangle$ (thus $x_m \in Z$). For each $k = 1, \ldots, m$, let B^k be the atomic sentence associated with arc $x_{k-1} x_k$. Let v be the valuation that assigns truth value T to each B^k ($k = 1, \ldots, m$) and truth value F to every other atomic sentence in $S(G)$. Then this valuation assigns truth value T to all the wffs in $\Gamma(G)$. □

Proof of Proposition 4.1: The proof of this proposition is very similar to that of proposition 3.1. First of all, from the definition of extensive form one can deduce the following. Let $g = \{y_1, \ldots, y_m\}$ be an information set such that $x_0 \notin g$, and p the corresponding player. For every $i=1,\ldots,m$ let x_{k_i} be the immediate predecessor of node y_i. Finally, let $c_1, c_2, \ldots, c_r$ be the choices at g. Then:

$$\tilde{\lambda}(c_1)\vee\tilde{\lambda}(c_2)\cup\ldots\cup\tilde{\lambda}(c_r)=\lambda(x_{k_1}y_1)\cup\lambda(x_{k_2}y_2)\cup\ldots\cup\lambda(x_{k_m}y_m) \qquad \text{(A.4)}$$

(where λ is the function – defined in section 3 – that associates with every arc η the set of terminal nodes reached by plays that contain η, and $\tilde{\lambda}$ is the function – defined in section 4 – that associates with every choice c the set of terminal nodes reached by plays that have an arc in common with c), and

$$\tilde{\lambda}(c_i)\cap\tilde{\lambda}(c_j)=\varnothing,\ \text{for all } i,j=1,\ldots,r,\ i\neq j \qquad \text{(A.5).}$$

Furthermore, if $\langle x_0 w_1, w_1 w_2, \ldots, w_{s-1} w_s\rangle$ is a path from the source to w_s, then

$$\lambda(x_0 w_1)\cap\lambda(w_1 w_2)\cap\ldots\cap\lambda(w_{s-1} w_s)=\lambda(w_{s-1} w_s) \qquad \text{(A.6).}$$

Now, let G be an extensive form. The proof that the wffs associated with the source are true in $\langle\Omega, f\rangle$ is the same as for proposition 3.1 (since a choice at the source is necessarily a single arc). Let $g=\{y_1,\ldots,y_m\}$ be an information set such that $x_0\notin g$, and p the corresponding player. Let $A_{p,k_1}, A_{p,k_2}, \ldots, A_{p,k_r}$ be the atomic sentences associated with the r choices at g. Then the fact that, for all $i,j=1,2,\ldots,r$, with $i\neq j$

$$(\tilde{\lambda}\circ\tilde{h})(\neg(A_{p,k_i}\wedge A_{p,k_j}))=Z$$

follows from (A.5), while the fact that

$$(\tilde{\lambda}\circ\tilde{h})(\Phi)=Z,$$

where Φ is the following wff associated with g (see section 4)

$$(A_{p,k_1}\vee\ldots\vee A_{p,k_r})\Leftrightarrow(B^{1,1}\wedge\ldots\wedge B^{1,n_1})\vee\ldots\vee(B^{m,1}\wedge\ldots\wedge B^{m,n_m}),$$

follows from (A.4) and (A.6). In fact, for every $i\in\{1,\ldots,m\}$, $(\tilde{\lambda}\circ\tilde{h})(B^{i,1}\wedge\ldots\wedge B^{i,n_i})=\lambda(x_{k_i}y_i)$, where x_{k_i} is the immediate predecessor of y_i. □

Proof of Proposition 4.2: The proof of this proposition mimics that of proposition 3.2. First we show that if v is a valuation that satisfies all the wffs in $\Gamma(G)$ then there is a unique play in G associated with it. Let $\mathscr{C}(v)$ be the set of choices of G whose corresponding atomic sentences are assigned truth value T by v. First of all, one choice at the source must be assigned truth-value T, otherwise the wff $(A_{p,k_1}\vee A_{p,k_2}\vee\ldots\vee A_{p,k_m})$, where $x_{k_1}, x_{k_2}, \ldots, x_{k_m}$ are the immediate successors of the source, would have truth-value F. Now let g be an arbitrary information set, p the corresponding player and c_i and c_j two different choices at g. Let $A_{p,k_i}=\tilde{h}^{-1}(c_i)$ and $A_{p,k_j}=\tilde{h}^{-1}(c_j)$ be the corresponding atomic sentences. Then it cannot be that $v(A_{p,k_i})=T$ and $v(A_{p,k_j})=T$ otherwise the wff $\neg(A_{p,k_i}\wedge A_{p,k_j})$, which belongs to $\Gamma(G)$, would have truth value F. Thus at most one choice at each information set is such that its corresponding wff is assigned truth-value T by v. Let $v(A_{p,k_i})=T$ and let node x_{k_i} belong to information set g. Then one choice at g must be such that the corresponding atomic sentence has truth-value T, otherwise the following wff associated with g would have truth-value F:

$$(A_{p,y_1} \vee A_{p,y_2} \vee \ldots \vee A_{p,y_r}) \Leftrightarrow A_{p,k_i}$$

where $A_{p,y_1}, \ldots, A_{p,y_r}$ are the atomic sentences associated with the choices at g. Thus following – at node $x_{k_i} \in g$ – the choice whose corrsponding atomic sentence has truth value T we reach a new node (and a new information set) and the same argument can be repeated, until a unique terminal node is reached.

Now we show the converse, namely that with every play in G we can associate a unique valuation v that satisfies all the wffs in $\Gamma(G)$. Fix an arbitrary play $\langle x_0 x_1, x_1 x_2, \ldots, x_{m-1} x_m \rangle$ (thus $x_m \in Z$). For each $k = 1, \ldots, m$, let B^k be the atomic sentence associated with the choice to which arc $x_{k-1} x_k$ belongs. Let v be the valuation that assigns truth value T to each B^k $(k = 1, \ldots, m)$ and truth value F to every other atomic sentence in $S(G)$. Then it is easy to see that this valuation assigns truth value T to all the wffs in $\Gamma(G)$. □

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