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MONOPOLY EQUILIBRIA AND CATASTROPHE THEORY

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Introduction

The possibility that agents may react discontinuously to continuous changes in their environment does not seem to have been sufficiently investigated in the literature. Intuition suggests that continuously changing causes should produce continuous effects. As a consequence, if a sudden jump occurs in some observed variable, one tends to assume that a discontinuous change must have taken place in one of the factors determining the level of that variable. With the advent of catastrophe theory (Thom, 1972; Zeeman, 1977) the occurrence of discontinuities in smoothly evolving systems has come to be recognized as a not unlikely and non-pathological phenomenon. As Dodgson (1982, p. 414) observes, "economics has been regarded as a likely area for the application of catastrophe theory but, despite interest among economists, few applications have emerged, and these have tended to be speculative in nature".[1] The general view among economists is probably that the types of discontinuities envisaged by catastrophe theory arise only in rather complex and artificial models, that is, that they are economically uninteresting. One of the purposes of this paper is to show that "catastrophes" can arise "naturally" in very simple and standard models.

We consider the case of a monopolist who sells to a large number of consumers who differ in their reservation price for the commodity. We provide a simple and useful characterization of profit maximization based on the hazard rate function associated with the distribution of reservation prices. This enables us to give a simple geometrical illustration of the nature of the discontinuities mentioned above.

The possibility of discontinuities arising from plurimodality of the profit function in the case of monopoly has been shown before in the literature (Walters, 1980; Formby *et al.*, 1982; Dodgson, 1982).[2] These authors, however, have restricted their attention to the case of "convex-kink demand curve" and have not gone beyond the analysis of the effects of changes in costs with a fixed demand curve. Furthermore, no direct application of the classification theorem of catastrophe theory has been given.[3] Finally, the possibility of "perverse" adjustments has not been investigated.

[1]A list of approximately forty references on applications of catastrophe theory to economics can be obtained from the author. For a lengthier discussion of catastrophe theory than the one given in this paper, see Bonanno and Zeeman (1986).

[2]The possibility of non-pathological discontinuous responses to changes in other players' strategies in *oligopoly* was pointed out by Roberts and Sonnenschein (1977).

[3]Dodgson's paper (1982) is somewhat an exception. The author, however, restricts his attention to the cusp catastrophe (see Figure 5(c)) and does not mention the other possibilities illustrated in Figure 5.

The paper is organized as follows. In the following Section we describe the model and in Section III we apply the classification theorem of catastrophe theory. Finally, in Section IV we give an example of "perverse" adjustments to changes in demand. Section V contains some final remarks and a conclusion.

II. The Model

We consider the case of a monopolist who produces an indivisible good (*e.g.* washing machines) and sells to a large number N of consumers. Each consumer buys at most one unit of the good and consumers differ in their reservation price for the product (the maximum price they are willing to pay for one unit of it). By a convenient choice of (monetary) units we can assume that the maximum reservation price is 1. Let

$$f\colon [0, 1] \rightarrow R \tag{1}$$

be the density function of reservation prices and

$$F\colon [0, 1] \rightarrow [0, 1] \tag{2}$$

be the corresponding cumulative distribution function.[4] Thus, for each $p \in [0, 1]$,

$$F(p) = \text{proportion of consumers with reservation price} \leq p \tag{3}$$

Therefore, $1 - F(p)$ is the proportion of consumers whose reservation price exceeds p and who, therefore, are willing to buy the product at price p. Hence the demand function facing the monopolist is given by

$$D(p) = N\,[1 - F(p)] \tag{4}$$

(Note that, since F is non-decreasing, D is non-increasing).

Let the cost function of the monopolist be given by

$$C(q) = K + c\,q \tag{5}$$

where $K \geq 0$ is fixed cost, $c \geq 0$ is marginal cost and q denotes output. The monopolist's profit function is therefore given by

$$\pi(p) = N\,(p - c)\,[1 - F(p)] - K \tag{6}$$

The first-order condition for profit maximization can be written as

$$p - c = \frac{1}{h(p)} \tag{7}$$

[4] A possible interpretation of the model of this Section is given in the Appendix.

where

$$h(p) = \frac{f(p)}{1 - F(p)} \tag{8}$$

is the *hazard rate* function (see, for example, Ross, 1984, p. 168) and gives the consumers with reservation price between p and $(p + dp)$ *as a proportion of the consumers who are willing to buy at price p*. Thus if the monopolist increases his price from p to $(p + dp)$, $100h(p)$ per cent of his customers will stop buying.[5] The characterization of profit maximization given by (7) allows us to draw some interesting conclusions.

First of all, the shape of the demand function D depends on the properties of f, since $D'' = -f'$. Thus D is strictly concave (strictly convex) if and only if f is increasing (decreasing) and is linear when f is constant (see Figure 1).[6]

In general, the profit function can have any shape. In fact, since the hazard rate function *uniquely* determines the distribution F (and thus the density f)[7], we can concentrate on the latter and generate any number of solutions to the first-order condition (7) by choosing the hazard rate function appropriately. An example is given in Figure 2.

[5]Hence the first-order condition (7) has a very intuitive interpretation. Let p_0 be the initial price and $D_0 = D(p_0) = N[1 - F(p_0)]$ the corresponding demand. Let the price be increased by Δp. Let α be the proportion of consumers who stop buying because of the price increase. Then

$$\alpha = \frac{D(p_0) - D(p_0 + \Delta p)}{D(p_0)} = \frac{F(p_0 + \Delta p) - F(p_0)}{1 - F(p_0)}$$

Therefore the change in profits is given by

$$\Delta\pi = (1 - \alpha)\,\Delta p\, D_0 - \alpha\, D_0\,(p_0 - c).$$

At a profit maximum, $\Delta\pi = 0$, *i.e.*

$$\Delta p + p_0 - c = \Delta p/\alpha$$

Now,

$$\lim_{\Delta p \to 0} \frac{\alpha}{\Delta p} = \lim_{\Delta p \to 0} \frac{F(p_0 + \Delta p) - F(p_0)}{\Delta p} \quad \frac{1}{1 - F(p_0)} =$$

$$= \frac{f(p_0)}{1 - F(p_0)} = h(p_0)$$

Thus a necessary condition for p_0 to maximise profits is

$$p_0 - c = 1/h(p_0).$$

[6]Thus D changes shape at the modes of f.

[7]See, for example, Ross (1984, p. 169).

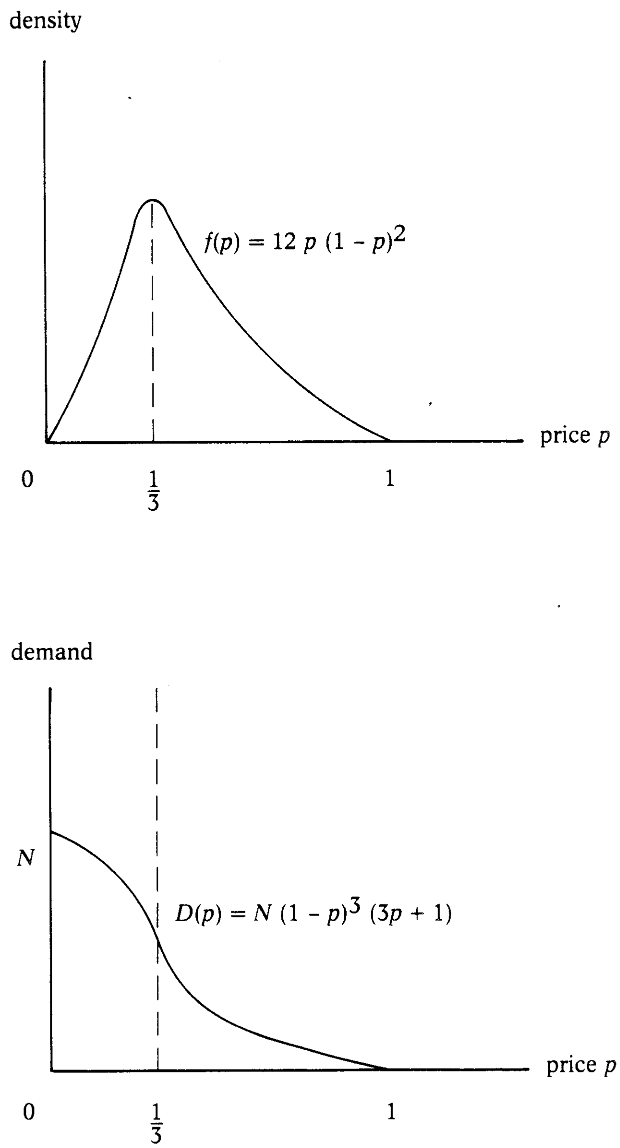

FIGURE 1

The density function is the beta distribution with parameters $\alpha = 2$ and $\beta = 3$. The corresponding c.d.f. is given by $F(p) = 3p^4 - 8p^3 + 6p^2$.

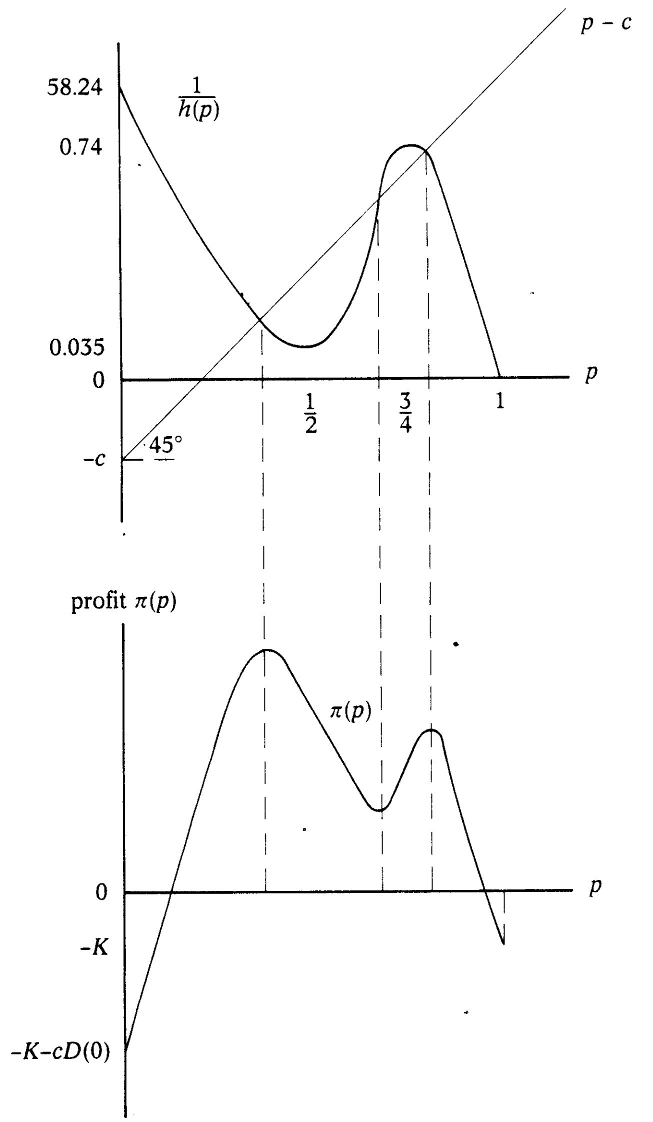

FIGURE 2

The hazard rate function is given by
$h(p) = 1/(-\ 96p^5 + 513.8p^4 - 996.6p^3 + 892.1p^2 - 371.5p + 58.2)$. The corresponding density and c.d.f. functions are given by

$$f(p) = h(p)\ \exp\{-\int_0^P h(u)du\} \text{ and } F(p) = 1 - \exp\{-\int_0^P h(u)du\} \text{ respectively.}$$

It can be shown[8] that if p satisfies the first-order condition (7), then

$$\frac{d}{dp}\,\frac{1}{h(p)} \leqq 0 \text{ implies } \pi''(p) \leq 0 \tag{9}$$

Thus the points at which the line of equation $z = p - c$ intersects the graph of the function $1/h(p)$, where the latter is decreasing, are local maxima of π.[9]

Secondly, the characterization given by (7) makes comparative statics results geometrically simple to analyze. Consider first *changes in costs with a fixed demand function* (*i.e.* a fixed hazard rate function h). As Figure 3 shows, an increase in marginal cost (represented by a parallel downward shift of the line of equation $z = p - c$) may give rise to a discontinuous increase in price. Thus even if the demand and cost functions are smooth, the profit-maximizing price – and the corresponding output – may be discontinous functions of the cost parameters. We shall show in the following Section that such discontinuities may be unavoidable.

Consider now a *change in demand with a fixed cost function*. Again, as shown in Figure 4, a smoothly evolving demand may give rise to large and discontinuous adjustments in the monopolist's price and output. In Section IV we shall give an example where the adjustment is somewhat surprising.

So far we have shown that a monopolist may react to small and continuous changes in the environment (demand and/or cost conditions) with large and discontinuous adjustments in price and output. In the next Section we apply catastrophe theory to classify the possible types of discontinuities.

III. The Classification Theorem of Catastrophe Theory

Let

$$\pi\colon P \times T_1 \times T_2 \to R \tag{10}$$

$$(p, t_1, t_2) \to \pi(p, t_1, t_2)$$

be the monopolist's profit function, where P, T_1 and T_2 are subsets of the real line. The decision variable is the price p, while t_1 and t_2 are parameters which define the environment. In order to highlight the difference between decision variable and

[8]Let p_0 be such that $\pi'(p_0) = 0$, *i.e.*

$$p_0 - c = 1/h(p_0) \tag{i}$$

Now, $\pi''(p_0) = N\{-2f(p_0) - (p_0 - c)f'(p_0)\}$. Thus $\pi''(p_0) > 0$ if and only if $-f(p_0) > f(p_0) + (p_0 - c)f'(p_0)$ and multiplying each side by $h(p_0)/[1 - F(p_0)]$ and using (i) this is equivalent to

$$-[h(p_0)]^2 > h'(p_0)$$

which cannot be true unless $h'(p_0) < 0$. Thus $\pi'(p_0) = 0$ and $h''(p_0) \geq 0$ implies $\pi''(p_0) \leq 0$. Note that the converse of (9) is not true.

[9]The points at which the line of equation $z = p - c$ is tangent to the graph of the function $1/h(p)$ are points of inflection of the profit function.

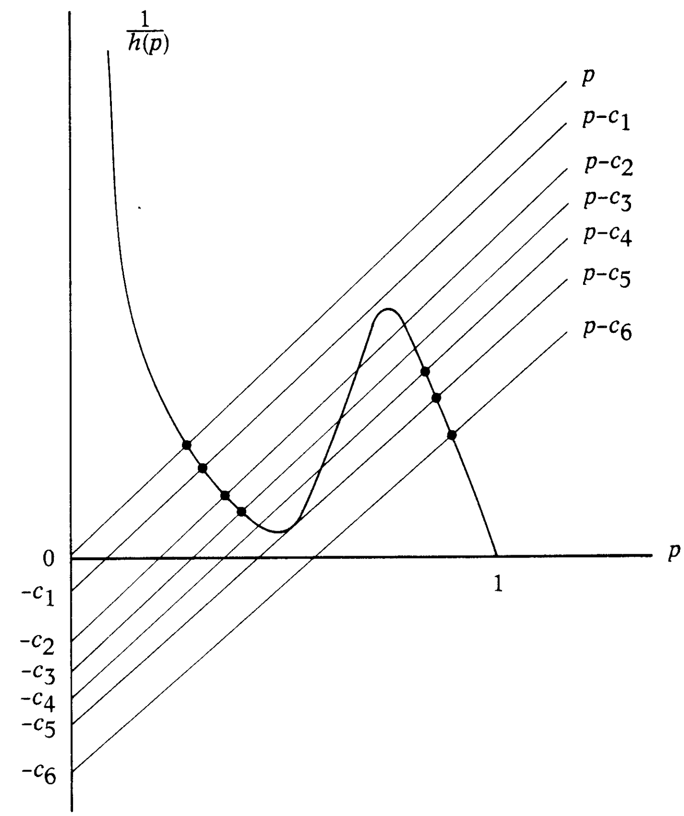

FIGURE 3
The hazard rate function is the same as in Figure 2. The abscissas of the large dots are the global maxima of the profit functions $\pi_c(p) = (p - c)D(p) - K$.

environment, we shall write $\pi_{t_1, t_2}(p)$. We shall mainly think of t_1 as a *demand parameter* and t_2 as a *cost parameter*. The demand and cost functions will in general be identified by more than one parameter each. Let demand be identified by k parameters and cost by m parameters. Then changes in demand and cost over time can be expressed by two functions

$$g_1\colon [0, 1] \to R^k \tag{11}$$

$$t_1 \to g_1(t_1)$$

and

$$g_2\colon [0, 1] \to R^m \tag{12}$$

$$t_2 \to g_2(t_2)$$

and we can consider t_1 as "the demand parameter" and t_2 as "the cost parameter". We want to analyze how the monopolist responds to changes in cost and/or demand.

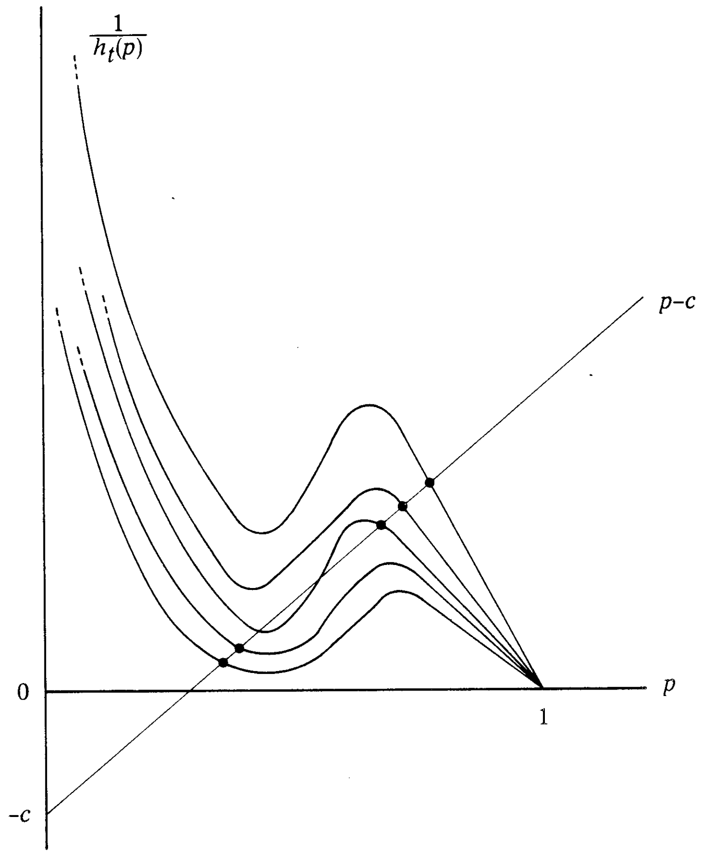

FIGURE 4
A smoothly evolving family of (the reciprocal of) hazard rate functions giving rise to a discontinuous adjustment in price. The abscissas of the large dots are the global maxima of the profit function $\pi_t(p) = (p - c)D_t(p) - K$.

Let

$$M(t_1, t_2) = \underset{p}{\operatorname{argmax}}\ \pi_{t_1, t_2}(p) =$$

$$= \{p \in P \ / \ \pi_{t_1, t_2}(p) \geq \pi_{t_1, t_2}(p'), \text{ for all } p' \in P\} \tag{13}$$

be the set of profit-maximizing prices when the environment is given by (t_1, t_2). In general, π_{t_1, t_2} will have a unique global maximum at a unique point $p \in P$ and so $M(t_1, t_2)$ will be a singleton. However, in special cases π_{t_1, t_2} may have two global maxima at the same level at two different points p_1, p_2 and in this case $M(t_1, t_2) = \{p_1, p_2\}$. Moreover, such special cases may be unavoidable, because perturbations one way may raise one of the two maxima to be the unique global maximum, while perturbations the other way may raise the other maximum, causing a discontinous jump as shown in Figure 5 (b, c). Since $T_1 \times T_2$ is two-dimensional, a further complexity can arise with three global maxima at the same level, as shown in Figure 5(d), but this is the worst possible case, as indicated by the theorem below. Let M^* (a subset of $P \times T_1 \times T_2$) be the graph of the correspondence $M(t_1, t_2)$:

$$M^* = \{(p, t_1, t_2) \in P \times T_1 \times T_2 \,/\, p \in M(t_1, t_2)\} \tag{14}$$

Finally, let W be the set of profit functions (10) which are smooth.[10] We endow W with the Whitney C^∞-topology (see Zeeman, 1977). We can now state the relevant classification theorem of elementary catastrophe theory due to René Thom.

THEOREM (Thom, 1972; Zeeman, 1977). *There exists an open dense subset V of W such that if $\pi \in V$, then the resulting equilibrium set M^* defined by* (14) *is a two-dimensional surface, which is locally equivalent*[11] *at each point to one of the graphs shown in Figure 5. Furthermore, the discontinuities of M^* are structurally stable, that is, they cannot be eliminated by small perturbations of π.*

Notice that in fact M^* is a surface-with-boundary, and that the boundaries occur whenever the the profit-maximizing price varies discontinously with the environment. In Figure 5 the discontinuities are indicated by vertical lines (which are not actually part of M^*).

Notice also that since V is dense in W, a function in W which does not belong to V can be approximated arbitrarily closely by a function in V and, furthermore, since V is open, if $\pi \in V$ then every small perturbation of π also belongs to V. Thus "almost all" profit functions belong to V.

Case (a) of Figure 5 is the intuitive situation that one would expect to observe: the profit-maximizing price varies continuously with variations in demand and costs.

Case (b) of Figure 5 is the counterintuitive situation of "unavoidable" discontinuous response: despite the fact that demand and cost conditions vary over a continuous range, we observe, essentially, only two extreme policies (high price-low output, low price-high output), rather than a continuum of policies. The two extreme policies are separated by a line (a curve) in the (t_1, t_2) space which is called the *Maxwell line*. This is the set of environments at which the profit function has two global maxima at the same level. As we move away from this line, only one of the two maxima remains a global maximum, while the other becomes a local, but not global, maximum (which of the two remains the global maximum depends on the direction of movement away from the Maxwell line). Therefore, demand and cost conditions represented by two points on either side of the Maxwell line may be so close as to be almost indistinguishable, yet they give rise to very different equilibrium prices, p_1 and p_2.

[10]The smoothness assumption is not a strong one, since every continuous function can be approximated arbitrarily closely by a smooth function. In other words, smooth functions are dense in the space of continuous functions: see Hirsh (1976, theorem 2.4, p. 47).

[11]"Local equivalence" means that for each $p \in P$ there is an embedding $T_1 \times T_2 \subset R^2$ underlying a projection $T_1 \times T_2 \times P \to R^2 \times \mathrm{R}$ that throws a neighbourhood of p in M^* diffeomorphically onto one of the graphs shown in Figure 5 (see Bonanno and Zeeman, 1986). Thus the way that M^* sits over $T_1 \times T_2$, reflecting the discontinuities of the optimal pricing policy, is portrayed qualitatively accurately by the way the graphs lie over the horizontal planes in Figure 5.

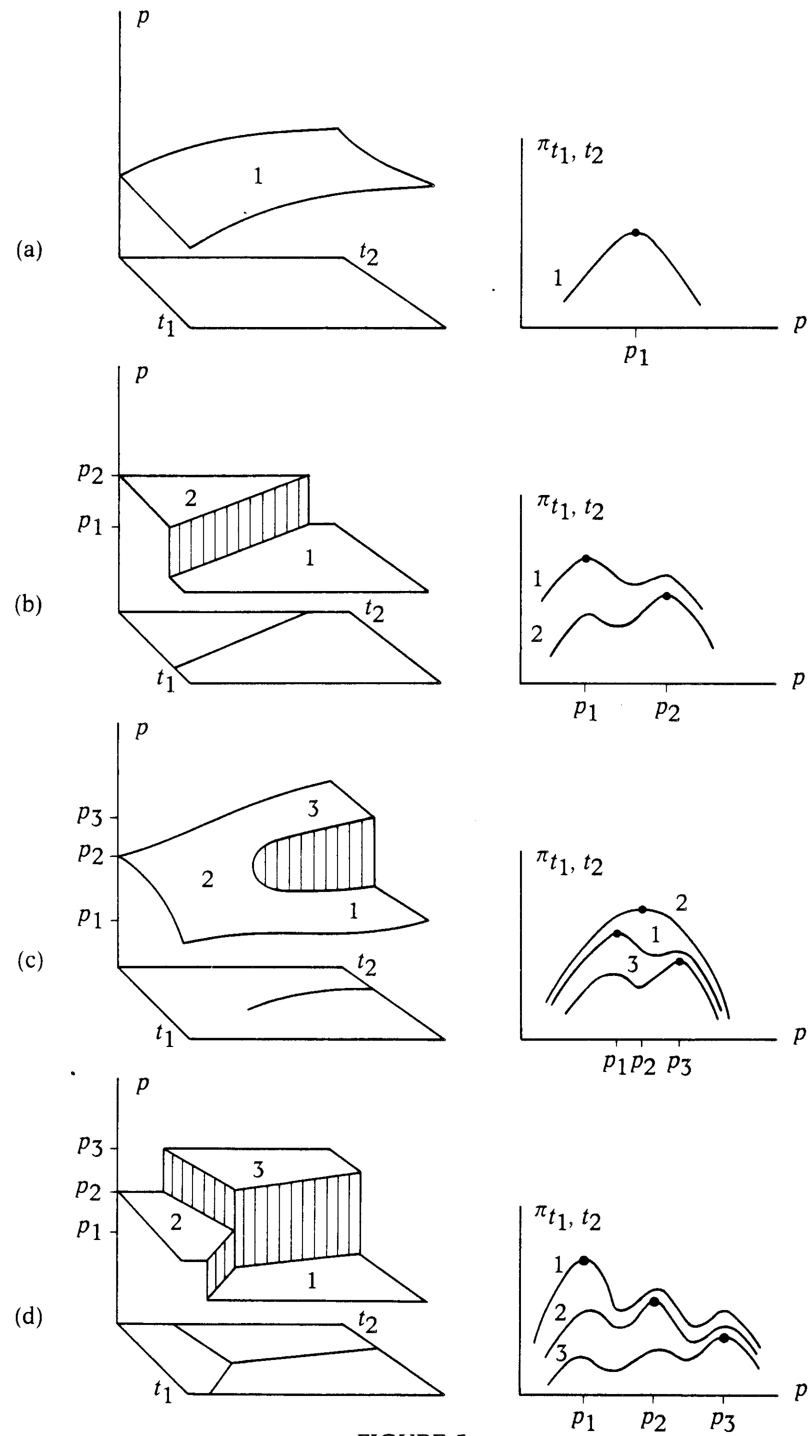

FIGURE 5

Case (c) of Figure 5 represents the "threshold of polarization": here the Maxwell line ends in a point which marks the onset of polarized pricing policy in a gradually changing environment. The graph arises from the cusp catastrophe (see Thom, 1972 and Zeeman, 1977). Before the threshold the monopolist reacts to changes in demand/cost conditions with a continuous spectrum of prices, but after the threshold the monopolist's pricing policy is split into two, p_1 and p_3. The middle choice p_2 is no longer observed.

Case (d) of Figure 5 is characterized by a Maxwell line which is Y-shaped and at the vertex of the Y the three regions representing essentially three different pricing policies meet.

In the next Section we give a simple example of the discontinuities shown in Figure 5, which furthermore has some surprising features.

IV. An Example

Since the way in which changes in costs can produce discontinuous adjustments in price and output has been shown geometrically in Section II and has been the object of investigation before in the literature (Walters, 1980; Formby *et al.*, 1982; Dodgson, 1982), we shall give an example where the discontinuities are associated with changes in demand. In order to simplify the exposition, we shall assume that the costs of production are zero or, equivalently, that the monopolist maximizes revenue; however, our results are entirely independent of this assumption. Let $D_t(p)$ be the demand function at time t. The example we consider is one where demand decreases continuously over time, that is

$$t'' > t' \text{ implies } D_{t''}(p) \leq D_{t'}(p) \text{ for all } p, \text{ and } D_{t''} \neq D_{t'} \tag{15}$$

In particular, we shall consider the situation where (15) is determined by a general impoverishment of consumers, which, however, is not uniform in the sense that *the rich classes shrink proportionately more than the poor classes* (this could be the effect of a tax, for example). An example is given in the table in Figure 6, where a proportion of consumers in each income bracket moves to the immediately preceding bracket, but at a decreasing rate: 50 per cent of the richest class, 37.5 per cent of the second richest, and 25 per cent and 12.5 per cent of the remaining two, respectively.

If this is the case, we would expect the monopolist to react by continously reducing his price over time. Instead we show that:

(i) the reduction in price may be discontinous,

(ii) somewhat more surprisingly the monopolist may not react at all to changes in demand (constant price over time),

(iii) even more surprisingly, the monopolist may react with a sudden *increase* in price followed by a policy of constant (high) price.

Since we shall translate the changes in the distribution of income illustrated in Figure 6 into changes in the distribution of reservation prices, we have in mind a model of the type described in the Appendix.

income ($/week)	% of consumers at time $t = 0$	at time $t = 1$
0 – 20	20%	22.5%
20 – 40	20%	22.5%
40 – 60	20%	22.5%
60 – 80	20%	22.5%
80 – 100	20%	10%

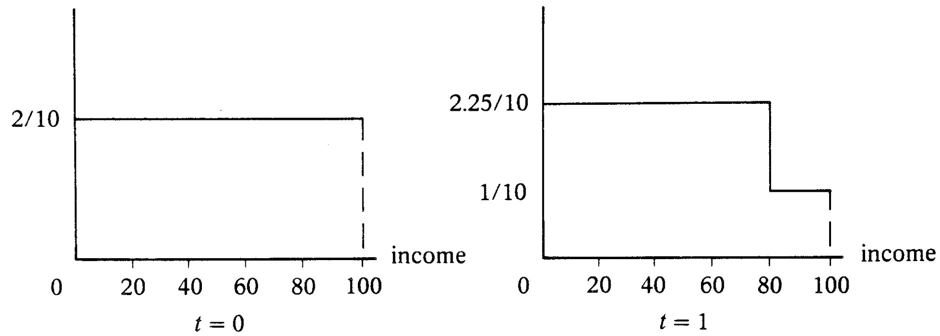

FIGURE 6
Between time 0 and time 1, 50% of the consumers in the bracket 80-100 have moved down to the bracket 60-80, 37.5% of those in the bracket 60-80 have moved down to the preceding bracket, 25% of those in the 40-60 bracket have moved to the 20-40 bracket and 12.5% of those in the 20-40 interval have moved down.

Consider the following two-parameter family of density functions[12]

$$f_{a,\,b}(p) = a \text{ if } 0 \leq p \leq l \qquad (16)$$

$$\qquad b \text{ if } l \leq p \leq 1$$

[12]The corresponding mean and variance are given by

$$\mu = \int_0^1 p f_{a,\,b}(p)dp = \frac{b + (1-b)l}{2}$$

and

$$v = \int_0^1 (p-\mu)^2 f_{a,\,b}(p)\,dp = \frac{b + (1-b)l^2}{3} - \mu^2$$

where

$$l = \frac{1 - b}{a - b} \tag{17}$$

Figure 7 illustrates the function $f_{a,\,b}$ for $0 \leq b \leq 1 \leq a$.

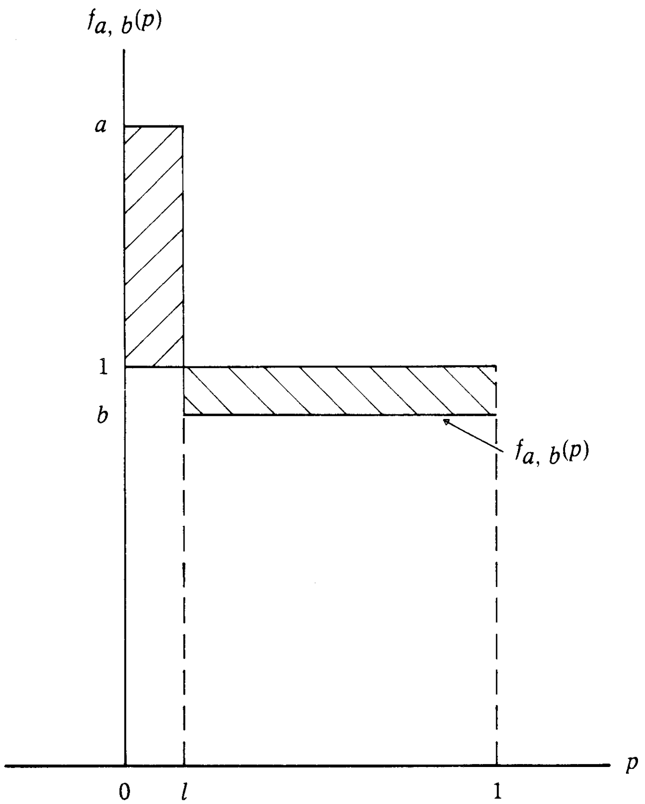

FIGURE 7
The density function $f_{a,\,b}(p)$. Given a and b, l is determined and equal to $(1 - b)/(a - b)$, so that the two shaded areas are equal.

Let $F_{a,\,b}$ be the corresponding family of cumulative distribution functions and let $D_{a,\,b}$ be the corresponding family of demand functions (given by (4)). Then

$$D_{a,\,b}(p) = \begin{cases} N(1 - ap) & \text{if } 0 \leq p \leq l \\ N(b - bp) & \text{if } l \leq p \leq 1 \end{cases} \tag{18}$$

Suppose the initial distribution (at time zero) is uniform, that is, at time zero

$$a_0 = b_0 = 1 \tag{19}$$

It is easy to check that (15) is satisfied if

$$a_{t''} \geq a_{t'} \text{ and } b_{t''} \leq b_{t'} \text{ and not both equal} \tag{20}$$

In Figure 8, starting from $a_0 = b_0 = 1$, a movement over time within the shaded area in the South, East or South-East direction implies that demand is decreasing over time due to a general impoverishment of consumers, as explained above.[13]

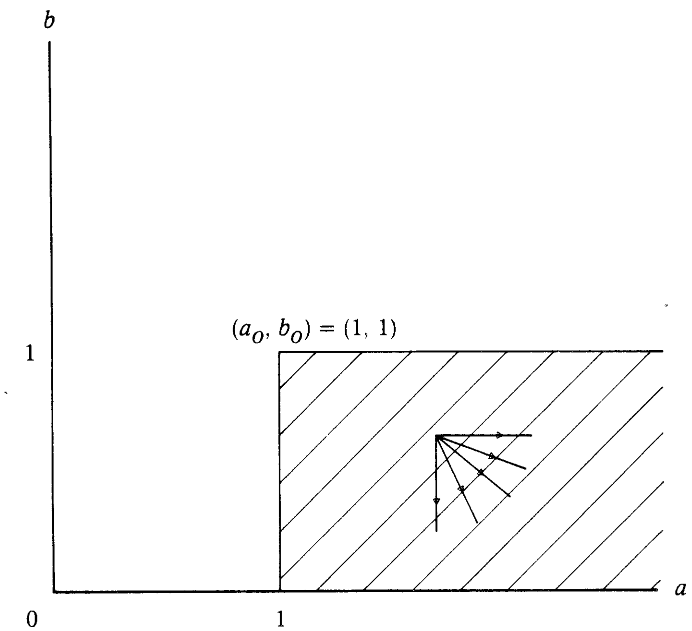

FIGURE 8
The arrows denote the directions along which demand decreases over time due to a general (non-uniform) impoverishment of consumers.

The monopolist's profit (and revenue) function is given by

$$\pi_{a,\, b}(p) = p\ D_{a,\, b}(p) \tag{21}$$

where $D_{a,\, b}$ is given by (18). Figure 9 shows the possible shapes of the function $\pi_{a,\, b}$.

It can be seen from Figure 9 that the set of optimal prices corresponding to the point (a, b) is given by

[13]It is easy to check that along such a path mean income μ and the variance v (see footnote 12) decrease over time.

$$
\begin{aligned}
M(a, b) &= \{1/2\} && \text{if } b > a^{-1} \\
&\quad \{1/2,\ 1/(2a)\} && \text{if } b = a^{-1} \\
&\quad \{1/(2a)\} && \text{if } b < a^{-1}
\end{aligned} \tag{22}
$$

Thus the Maxwell line is the curve of equation

$$
b = a^{-1} \qquad (a \geq 1) \tag{23}
$$

As explained in Section III, the Maxwell line is the set of environments (a, b) at which the profit function $\pi_{a,\,b}(p)$ has two global maxima at the same level (cf. Figure 9(b)). Thus, as the Maxwell line is crossed, the profit-maximizing price jumps discontinuously (cf. Figure 9(a, c)). The Maxwell line has a vertex at (1, 1) (outside the shaded area, the set $M(a, b)$ is a singleton and therefore the monopolist's reaction is continuous). Figure 10 reproduces

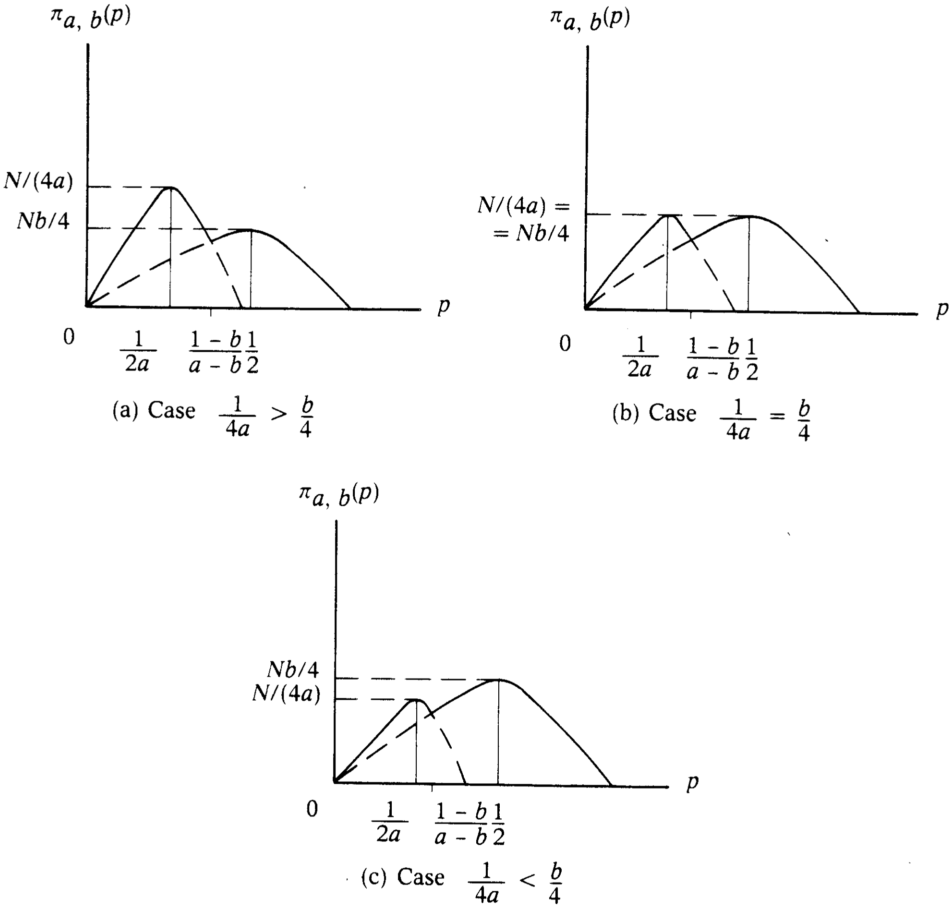

FIGURE 9

Figure 8 with the addition of the Maxwell line. Furthermore, we have denoted the profit maximizing price by p^* (thus $p^*(a, b) = M(a, b)$ when the latter is a singleton, *i.e.* at points not on the Maxwell line). Consider the three paths in Figure 10, all satisfying condition (15) (demand continuously decreasing over time due to a reduction in consumers' income which affects the rich proportionately more than the poor).

Along *path* (1) in Figure 10 the monopolist reacts discontinuously to the decrease in demand: at first he does not change his price ($p^* = 1/2$) and then he suddenly adjusts his price downward *discontinuously* (when the Maxwell line is crossed) and from then on follows with a continuous reduction in price over time.

Path (2) in Figure 10 has the surprising feature that, although demand is constantly decreasing over time, the monopolist does not react at all: his price remains constant over time and equal to 1/2.

Path (3) in Figure 10 shows an even more surprising feature: at first the monopolist reacts to continuous reductions in demand by continuously reducing his price and then, suddenly, he *increases* his price discontinuously and maintains a high price from then on. This behaviour is even more surprising considering the fact that the richer classes are shrinking at a higher rate than the less rich ones. The reason for this surprising reaction is that the reduction in demand due to the general (non-uniform) impoverishment of consumers creates a conflict of policy for the monopolist: the monopolist can either reduce his price in order to maintain as many of his original customers as possible or he can try to

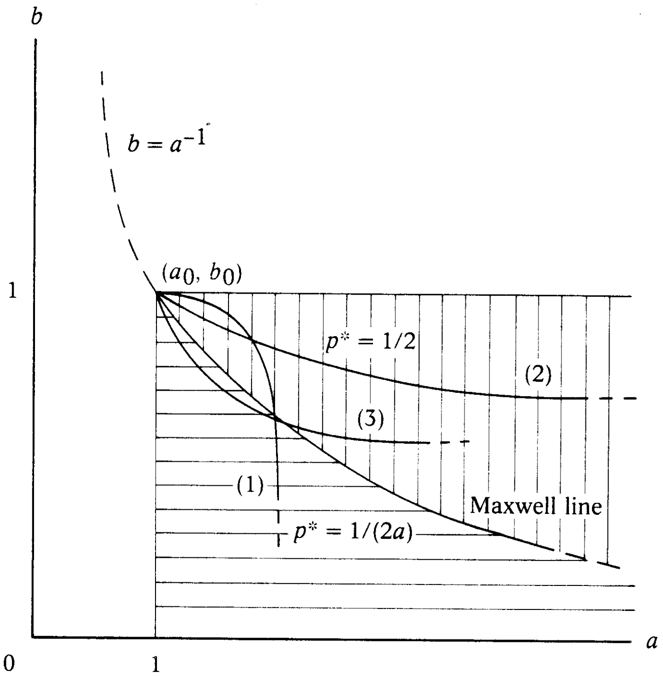

FIGURE 10
The arrows denote the evolution of demand over time along three alternative paths (1)-(3)

maintain a high profit margin, thereby forcing most of his original customers to drop out. The first policy requires continuous reductions in price and a point comes where the price has to be set so low that the alternative policy of selling an exclusive good to the smaller and smaller rich class now becomes more profitable. Hence the discontinuous upward jump in price. Note that intermediate policies are not profitable: the monopolist's pricing policy is polarized between a low price-high output one and a high price-low output one.

V. Remarks and Conclusions

In the example given above the demand function $D_{a,\,b}$ is a continuous but not smooth function of p. This implies that also the profit function $\pi_{a,\,b}$ is continuous but not smooth. We chose this example because of its simplicity. However, the kink in the demand function occurs at a local minimum of the profit function (see Figure 9) and the minima of the profit function are irrelevant from our point of view. Therefore we can choose a smooth approximation of (18) (cf. footnote 10) and obtain the same results. We can then invoke catastrophe theory to assert that all sufficiently close smooth approximations of (18) would yield the same qualitative results.

In the example of the preceding Section we only had changes in demand over time. If we now allow also costs to vary (when the cost function is given by (5)), then by the classification theorem of catastrophe theory we know that the set of optimal pricing policies must look (locally) like Figure 5 (b or c). It is worth noting, however, that the classification theorem as stated in Section III refers only to two generic parameters, not necessarily a cost and a demand parameter. Thus we can apply it directly to (a smooth approximation of) the model of section IV where the parameters are a and b. We can then conclude that the *global* shape of M^* is equivalent to that of Figure 5(c).

The classification theorem stated in Section III also tells us that a more complicated behaviour may arise, namely that illustrated in Figure 5(d). Such behaviour could arise, for example, with a family of three-step density functions, rather than the more simple two-step functions considered in Section IV. In such a situation, the monopolist would essentially be facing three classes of consumers: the poor, the moderately well-off and the rich. As consumers drift from one class to the next, the monopolist might find it optimal to change his price discontinuously, so as to sell mostly to one of the three classes.[14]

Without the classification theorem of catastrophe theory – in particular that part which tells us that the discontinuities arise generically and are structurally stable – the example of section IV would be less interesting. Catastrophe theory, however, justifies the claim that situations of this sort are not unlikely and, if they occur, they do so in a structurally stable way, that is, they cannot be eliminated by changing the specification of the model slightly.

[14]The case of Figure 5(d) is best understood as a section of the butterfly catastrophe. For more details, see Bonanno and Zeeman (1986).

Appendix

One way of thinking about the model of Section II is as follows. Suppose the incomes of the monopolist's potential customers lie in the range [O, E] and let

$$f^* : [0, E] \rightarrow R \qquad \text{(i)}$$

be the density function of income with corresponding c.d.f. given by

$$F^* : [0, E] \rightarrow [0, 1] \qquad \text{(ii)}$$

where

$$F^*(y) = \int_0^y f^*(u)du \qquad \text{(iii)}$$

Assume that each consumer is willing to spend at most a fraction $s \in [0, 1]$ of her income on the good (s does not vary across consumers). Thus a consumer with income y has a reservation price for the good equal to sy, that is, she will buy the good if and only if the price is less than sy. Normalize income so that $sE = 1$. Let

$$f, F : [0, 1] \rightarrow R \qquad \text{(iv)}$$

be defined by

$$f(p) = (1/s)f^*(p/s) \qquad \text{(v)}$$

and

$$F(p) = F^*(p/s) \qquad \text{(vi)}$$

Then (v) and (vi) correspond to (1) and (2).

The following situation, based on Gabszewicz and Thisse (1979), provides a concrete example. Consumers have identical preferences but different incomes. Let $J = \{1, 2, \ldots, N\}$ be the set of consumers ordered according to increasing income. Thus consumer j+1 is richer than consumer j. The income of consumer j is denoted by $y(j)$. If consumer j does not buy the product, her utility is given by

$$U(0, y(j)) = U_0 y(j), \quad U_0 > 0 \qquad \text{(vii)}$$

while if she buys one unit of the product at price p, her utility is given by

$$U(1, y(j)-p) = U_1 [y(j) - p], U_1 > U_0 \qquad \text{(viii)}$$

Then the reservation price of consumer j, denoted by $r(j)$, is that number which solves the following equation with respect to p:

$$U(0, y(j)) = U(1, y(j) - p) \qquad \text{(ix)}$$

Thus, using (vii) and (viii),

$$r(j) = s\ y(j) \qquad \text{(x)}$$

where

$$0 < s = (U_1 - U_0)/U_1 < 1 \qquad \text{(xi)}$$

Then (after a normalization of income which ensures that $sy(N) = 1$) for each $p \in [0, 1]$

$$F(p) = \#\{j \in J \ / \ y(j) \leq p/s\}\ (1/N) \qquad \text{(xii)}$$

(where $\#A$ denotes the number of elements in the set A). If the number of consumers is very large, the step function $F(p)$ can be approximated by a continuous function (then $f = F'$ almost everywhere).

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