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Memory of Past Beliefs and Actions

Abstract. Two notions of memory are studied both syntactically and semantically: memory of past beliefs and memory of past actions. The analysis is carried out in a basic temporal logic framework enriched with beliefs and actions.

Keywords: epistemic logic, temporal logic, memory, action, recall.

1. Introduction

Memory is usually identified with the retention of past knowledge (see, for example, [8] and [9]). Thus to model memory one needs two components: time and epistemic states. Semantically, this can be done with a set of instants T and two binary relations on it: a temporal relation $\prec$ (the interpretation of $t \prec x$ is that instant t precedes instant x) and an equivalence relation $\sim$ ($t \sim x$ means that the individual cannot distinguish between t and x). Graphically we shall represent the relation $\prec$ by drawing an arrow from t to x if $t \prec x$ and by enclosing t and x in a rounded rectangle if $t \sim x$. For example, Figure 1a illustrates a situation where at x the individual remembers what he knew previously (i.e. at t), while in the situation represented in Figure 1b at x the individual has forgotten what he knew before (he does not remember if in the past he was at t or at t').

The concept of memory can be captured by the following property, which is illustrated in Figure 2.

If $t \prec x$ and $x \sim x'$ then there exists a t' such that $t \sim t'$ and $t' \prec x'$. (Π1)

Property (Π1) corresponds to the following axiom:

$$PK\phi \to KP\phi \qquad (X1)$$

where P is the past operator ($P\psi$ means that some time in the past it was the case that ψ) and K is the knowledge operator ($K\psi$ means that the individual knows that ψ). Thus ($X1$) says that if, at some time in the past, the individual knew that ϕ, then she knows now that some time in the past it was the case that ϕ. Throughout the paper we shall use the Greek letter

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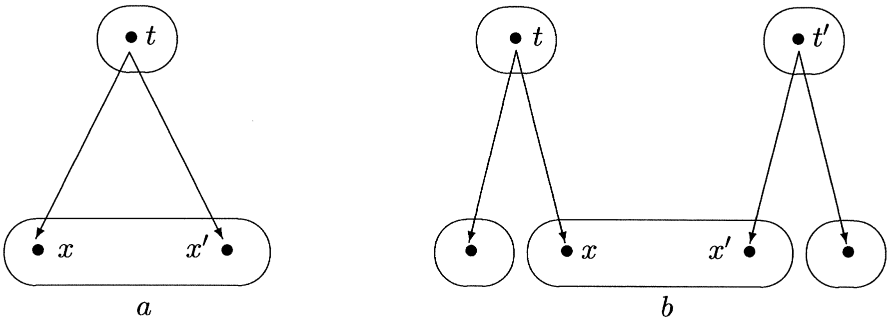


Figure 1.

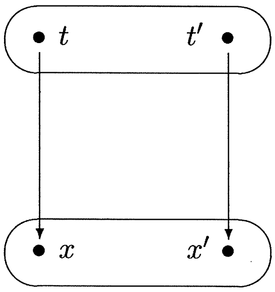


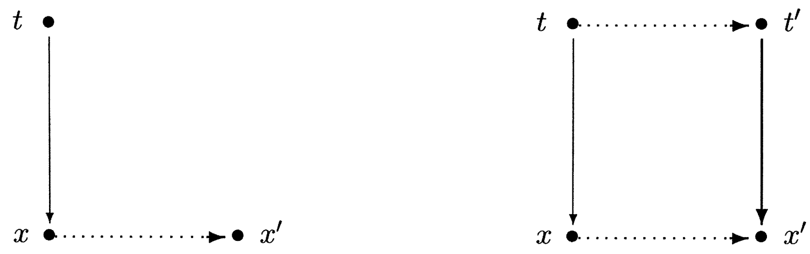
Figure 2.

Π to name properties of frames and the letter X to name axioms (we avoid P and A because the former is used for the past operator and the latter to denote actions).

In this paper we take a more general point of view and model memory as the *recollection of past epistemic states*, rather than knowledge in particular. For example, the individual might have held incorrect beliefs in the past and subsequently learned of her mistake. As long as she correctly remembers her past beliefs we would still claim that she has memory. In other words, it seems that *correctness of beliefs is not a property which is inherent to the notion of memory.* Accordingly, instead of an equivalence relation $\sim$ we will consider a more general binary relation $\mathcal{B}$ and interpret $t\mathcal{B}x$ to mean that at state t the individual considers state x possible. In this more general setup it may seem that the notion of memory can be captured by the following adaptation of property (Π1):

$$\text{If } t \prec x \text{ and } x\mathcal{B}x' \text{ then there exists a } t' \text{ such that } t\mathcal{B}t' \text{ and } t' \prec x'. \quad (\Pi 2)$$

This property is illustrated in Figure 3, where – as before – a continuous arrow from t to x denotes that $t \prec x$, whereas a dotted arrow from t to t' denotes that $t\mathcal{B}t'$.



this configuration implies this configuration

Figure 3.

It will be shown later that (Π2) corresponds to the following counterpart to axiom $(X1)$:

$$PB\phi \to BP\phi \qquad (X2)$$

where B is the belief operator ($B\psi$ means that the individual believes that ψ). Thus $(X2)$ says that if, at some time in the past, the individual believed that ϕ, then she believes now that some time in the past it was the case that ϕ.[1]

Is $(X2)$ the appropriate axiom to capture the notion of memory? We maintain that it is not. Consider, for example, the following situation. David's daughter was on the telephone with her friend Ann. From the conversation he overhears, David believes that his daughter is talking to her boyfriend Bob. When his daughter leaves the house, David presses the redial button on the telephone with the intention of asking Bob to leave his daughter alone. This situation is shown in Figure 4. State t represents the initial situation where David erroneously believes that after he presses the redial button, Bob will answer the phone. As a matter of fact, Ann answers. Let ϕ be the proposition "after the redial button is pressed, Bob answers the phone". Then at x it is true that in the past (i.e. at t) David believed that ϕ, written $x \models PB\phi$. However, once Ann answers the phone, David realizes his mistake and no longer believes that some time in the past pressing the redial button would lead to Bob answering the phone. That is, at x it is not the case that David believes that at some time in the past it was the case that ϕ, written $x \not\models BP\phi$. Thus axiom $(X2)$ is not satisfied at x. However,

[1] A referee pointed out that an alternative intuition behind $(X1)$ and $(X2)$ is that of "information increase" or "no revision".

at x David correctly remembers his past beliefs ($x \models BPB\phi$), although he now realizes that they were erroneous. We conclude that ($X2$) cannot be taken as the axiom that captures the notion of memory of past beliefs.

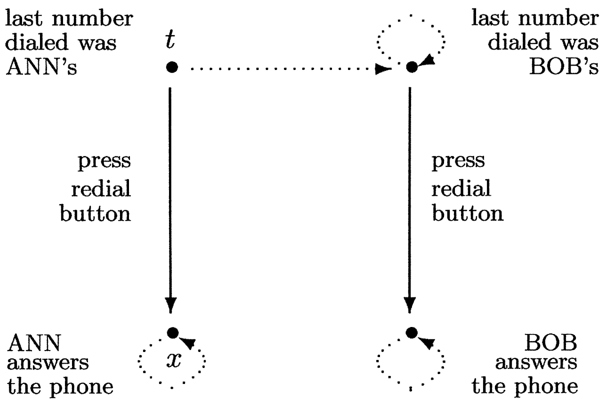


Figure 4.

We maintain that the appropriate axiom for memory is the following:

$$PB\phi \rightarrow BPB\phi \qquad (X_{mem})$$

which says that if, at some time in the past, the individual believed that ϕ, then she believes now that some time in the past she *believed* that ϕ. Axiom (X_{mem}) and its relationship to ($X2$) are studied in the next section.

Recollection of past epistemic states is just one type of memory. Remembering what one did in the past is another important aspect of memory. Indeed, as shown by Piccione and Rubinstein in [10], puzzles and paradoxes arise in decision situations where individuals have imperfect memory of their past actions.[2] In Section 3 we study the notion of memory of past actions by adding actions to epistemic temporal logic and discuss the relationship between memory of past actions and memory of past beliefs. Section 4 concludes.

2. Memory of past beliefs

We consider frames $\langle T, \prec, \mathcal{B} \rangle$, where T is a set of instants or states and $\prec$ and $\mathcal{B}$ are binary relations on T: $\prec$ is the *temporal precedence* relation, while $\mathcal{B}$

[2] An entire issue of *Games and Economic Behavior* (Vol. 20, 1997) has been devoted to the debate concerning the proper way of modeling decision-making under imperfect recall.

is the *belief* relation. Thus $t_1 \prec t_2$ means that instant t_1 precedes instant t_2, while $t_1 \mathcal{B} t_2$ means that at t_1 the individual considers t_2 possible. For the sake of generality, no properties are imposed on the temporal precedence relation $\prec$, so that the logic that we consider is basic temporal logic.[3] Whenever there are results that require more structure (e.g. a tree-like structure for $\prec$) the assumed properties will be stated explicitly. The same applies to the belief relation $\mathcal{B}$: no properties are assumed unless explicitly stated, so that the logic of belief that we consider is the normal system K (see [4][p. 115]; in particular, we do not assume positive or negative introspection, nor the truth axiom).

On the syntactic side, we consider a propositional language with three modal operators: the temporal operators G and H and the belief operator B. The intended interpretation is:

$G\phi$: "it is **G**oing to be the case at every future time that ϕ"
$H\phi$: "it **H**as always been the case that ϕ"
$B\phi$: "the individual **B**elieves that ϕ".

The formal language is built in the usual way from a countable set S of atomic propositions, the connectives $\neg$ (for "not") and $\vee$ (for "or") and the modal operators.[4] Let $F\phi \overset{def}{=} \neg G \neg \phi$ and $P\phi \overset{def}{=} \neg H \neg \phi$. Thus the interpretation is:

$F\phi$: "at *some* **F**uture time it will be the case that ϕ"
$P\phi$: "at *some* **P**ast time it was the case that ϕ".

Given a frame $\langle T, \prec, \mathcal{B} \rangle$ one obtains a *model based on it* by adding a function $V : S \to 2^T$ (where 2^T denotes the set of subsets of T) that associates with every atomic proposition $q \in S$ the set of states at which q is true. Truth of a formula ϕ at a state t, denoted by $t \models \phi$, is defined inductively as follows:

if q is an atomic proposition, $t \models q$ if and only if $t \in V(q)$,
$t \models \neg\phi$ if and only if $t \nvDash \phi$
$t \models \phi \vee \psi$ if and only if either $t \models \phi$ or $t \models \psi$,
$t \models G\phi$ if and only if $t' \models \phi$ for all t' such that $t \prec t'$,
$t \models H\phi$ if and only if $t'' \models \phi$ for all t'' such that $t'' \prec t$,
$t \models B\phi$ if and only if $t' \models \phi$ for all t' such that $t\mathcal{B}t'$.

Thus $G\phi$ ($H\phi$) is true at state t if and only if ϕ is true at *every* successor (predecessor) of t, while $F\phi$ ($P\phi$) is true at t if and only if ϕ is true at *some*

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[3] See, for example, [1, 3] and [6].

[4] See, for example, [4]. The connectives $\wedge$ (for "and") and $\to$ (for "if ... then") are defined as usual: $\phi \wedge \psi \overset{def}{=} \neg(\neg\phi \vee \neg\psi)$ and $\phi \to \psi \overset{def}{=} \neg\phi \vee \psi$.

successor (predecessor) of t. Furthermore, $B\phi$ is true at state t if and only if ϕ is true at every state that the individual considers possible at t. By means of models one can capture either an evolving world and the extent to which the individual learns about it, or a constant world in which the only thing that changes is the epistemic state of the individual. An example of the latter is an archaeologist's reaction to the discovery of new clues about events that occurred in the distant past.

We denote by $||\phi||$ the truth set of formula ϕ, that is, $||\phi|| = \{t \in T : t \models \phi\}$.

A formula ϕ is *valid in a model* if $t \models \phi$ for all $t \in T$, that is, if ϕ is true at every state. A formula ϕ is *valid in a frame* if it is valid in every model based on it.

Finally, we say that a property of a frame (e.g. property (Π2) above) is *characterized by* an axiom if the axiom is valid in every frame that satisfies the property and, conversely, if whenever the axiom is valid in a frame then the frame satisfies the property.

PROPOSITION 1. *Property* (Π2) *(see Section 1) is characterized by either of the following axioms:*

$(X2)\quad PB\phi \rightarrow BP\phi$

$(X2')\quad B\phi \rightarrow GBP\phi.$

$(X2)$ says that if in the past the individual believed ϕ, then she believes now that some time in the past it was the case that ϕ. While $(X2)$ is backward-looking, $(X2')$ is forward looking: it says that if the individual believes ϕ now, then at every future time she will believe that some time in the past it was the case that ϕ.[5]

PROOF. Assume (Π2). We show that both $(X2)$ and $(X2')$ are valid. For $(X2)$: suppose that $x \models PB\phi$. Then there exists a t such that $t \prec x$ and $t \models B\phi$. Fix an arbitrary x' such that $x\mathcal{B}x'$. By (Π2) there exists a t' such that $t\mathcal{B}t'$ and $t' \prec x'$. Since $t \models B\phi$, $t' \models \phi$. Thus $x' \models P\phi$ and $x \models BP\phi$. For $(X2')$: suppose that $t \models B\phi$. Fix arbitrary x and x' such that $t \prec x$ and $x\mathcal{B}x'$. By (Π2) there exists a t' such that $t\mathcal{B}t'$ and $t' \prec x'$. Since $t \models B\phi$, $t' \models \phi$. Thus $x' \models P\phi$ and $x \models BP\phi$ and $t \models GBP\phi$.

To prove the converse, assume that (Π2) does not hold, that is, there exist $t, x, x' \in T$ such that:

$$t \prec x, \quad x\mathcal{B}x', \text{ and} \qquad \text{(i)}$$

[5] Syntactically $(X2')$ can be derived from $(X2)$ by means of the valid principle (theorem of basic temporal logic) $\phi \rightarrow GP\phi$, and $(X2)$ can be derived from $(X2')$ by means of the valid principle $PG\phi \rightarrow \phi$.

$$\forall t' \in T, \text{ if } t' \prec x' \text{ then not } t\mathcal{B}t'. \tag{ii}$$

We want to show that both $(X2)$ and $(X2')$ can be falsified. Let q be an atomic proposition and construct a model where $||q|| = \{t' \in T : t\mathcal{B}t'\}$. Then

$$t \models Bq. \tag{iii}$$

For every t' such that $t' \prec x'$, by (ii), $t' \nvDash q$. Thus $x' \nvDash Pq$. Hence, by (i),

$$x \nvDash BPq. \tag{iv}$$

By (i) and (iii), $x \models PBq$. This, together with (iv), falsifies $(X2)$ at x. By (iv) and (i), $t \nvDash GBPq$. This, together with (iii), falsifies $(X2')$ at t. ■

As argued in the introduction, axiom $(X2)$ (or its equivalent counterpart $(X2')$) cannot be taken to correctly capture the notion of memory of past beliefs. The frame of Figure 4 violates property $(\Pi 2)$, and hence axiom $(X2)$ is not valid in it, and yet it *is* the case that the individual correctly remembers his past beliefs (although he now realizes that he was mistaken in those beliefs). We maintain that the following property does capture the notion of memory:

If $t \prec x$ and $x\mathcal{B}x'$ then there exists a $t' \in T$ such that (1) $t' \prec x'$ and (2) $\forall t'' \in T$, if $t'\mathcal{B}t''$ then $t\mathcal{B}t''$. (Π_{mem})

PROPOSITION 2. *Property (Π_{mem}) is characterized by either of the following axioms:*

(X_{mem}) $\quad PB\phi \rightarrow BPB\phi$
(X'_{mem}) $\quad B\phi \rightarrow GBPB\phi.$

(X_{mem}) is the backward-looking version of memory: it says that if, at some time in the past, the individual believed ϕ then she believes now that at some time in the past she believed ϕ. On the other hand, (X'_{mem}) is forward-looking: it says that if the individual believes ϕ now, then at every future time she will believe that at some time in the past she believed ϕ.[6]

PROOF. Assume (Π_{mem}). We show that both (X_{mem}) and (X'_{mem}) are valid. For (X_{mem}): suppose that $x \models PB\phi$. Then there exists a t such that $t \prec x$ and $t \models B\phi$. Fix an arbitrary x' such that $x\mathcal{B}x'$. By (Π_{mem}) there exists a t' such that $t' \prec x'$ and $\forall t'' \in T$, if $t'\mathcal{B}t''$ then $t\mathcal{B}t''$. Fix an arbitrary t'' such

[6] As in the case of $(X2')$ and $(X2)$, (X'_{mem}) can be derived from (X_{mem}) by using $\phi \rightarrow GP\phi$, and (X_{mem}) can be derived from (X'_{mem}) by using $PG\phi \rightarrow \phi$.

that $t'\mathcal{B}t''$ (if there is no such t'' then $t' \models B\phi$ vacuously). By (Π_{mem}) $t\mathcal{B}t''$. Thus, since $t \models B\phi$, $t'' \models \phi$. Hence $t' \models B\phi$ and $x' \models PB\phi$ and, therefore, $x \models BPB\phi$. For (X'_{mem}): suppose that $t \models B\phi$. Fix arbitrary x and x' such that $t \prec x$ and $x\mathcal{B}x'$. By (Π_{mem}) there exists a t' such that $t' \prec x'$ and $\forall t'' \in T$, if $t'\mathcal{B}t''$ then $t\mathcal{B}t''$. It follows from $t \models B\phi$ and the fact that $t'\mathcal{B}t''$ implies $t\mathcal{B}t''$, that $t' \models B\phi$. Thus $x' \models PB\phi$ and $x \models BPB\phi$. Hence $t \models GBPB\phi$.

To prove the converse, assume that (Π_{mem}) does not hold, that is, there exist $t, x, x' \in T$ such that

$$t \prec x, \quad x\mathcal{B}x', \text{ and} \tag{v}$$

$$\forall t' \in T, \text{ if } t' \prec x' \text{ then } \exists t'' \text{ such that } t'\mathcal{B}t'' \text{ and not } t\mathcal{B}t''. \tag{vi}$$

We want to show that both (X_{mem}) and (X'_{mem}) can be falsified. Let q be an atomic proposition and construct a model where $||q|| = \{t' \in T : t\mathcal{B}t'\}$. Then

$$t \models Bq. \tag{vii}$$

For every t' such that $t' \prec x'$, by (vi) $t' \not\models Bq$ and therefore $x' \not\models PBq$. Hence, by (v),

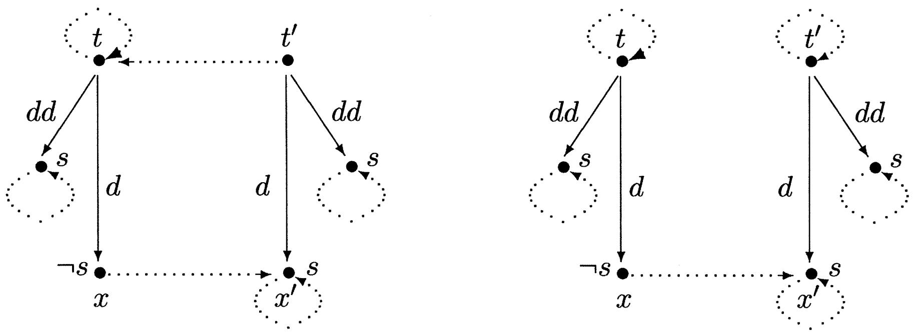
$$x \not\models BPBq. \tag{viii}$$

By (v) and (vii), $x \models PBq$. This, together with (viii), falsifies (X_{mem}) at x. By (viii) and (v), $t \not\models GBPBq$. This, together with (vii), falsifies (X'_{mem}) at t. ■

The frame of Figure 4 satisfies property (Π_{mem}) and therefore validates both (X_{mem}) and (X'_{mem}). In that example the individual correctly remembers his past beliefs, while at the same time realizing that they were mistaken. His realization comes after taking an action (pressing the redial button) that leads him to learn the truth. Figures 5*a* and 5*b* illustrate the opposite situation, where the individual initially holds correct beliefs and later, as a consequence of taking an action (drinking), starts believing something which is false.[7] At state t the individual is sitting at a bar and correctly believes that if he drinks it will be unsafe to drive. At state x, after drinking, the individual remembers that he drank and yet mistakenly believes that it is safe to drive. In Figure 5*a* at state x the individual correctly remembers his old belief (that it would be unsafe to drive) but does not take it seriously

[7] I learned this example from Johan van Benthem. The first version of this paper contained only Figure 5*b*. A referee pointed out that (X_{mem}) is compatible with switching from correct to incorrect beliefs and suggested the alternative scenario of Figure 5*a*.

anymore, while in Figure 5b at state x the individual is mistaken, not only about the safety of driving, but also in his recollection of his past belief (he believes that he previously believed, as he does now, that it would be safe to drive). While the frame of Figure 5a validates (X_{mem}), that of Figure 5b does not. In fact, let q be an atomic proposition that is true at t and false at t': $t \models q$ and $t' \models \neg q$. Then $t \models Bq$ and therefore $x \models PBq$. For every formula ϕ, $x \models B\phi$ if and only if $x' \models \phi$. Since $t' \models \neg q$, $t' \not\models Bq$. Hence $x' \not\models PBq$ and $x \not\models BPBq$. Thus (X_{mem}) is falsified at x.



a correctly remembering *b* being mistaken

Figure 5. Action d means 'drink', dd denotes 'don't drink' and proposition s means 'safe to drive'.

What is the relationship between axioms $(X2)$ and (X_{mem})? There are classes of frames where the two axioms are equivalent, that is, validity of one implies validity of the other.

PROPOSITION 3. *In the class of frames* $\langle T, \prec, \mathcal{B}\rangle$ *where the belief relation* $\mathcal{B}$ *is reflexive and transitive,*[8] (X_{mem}) *is valid if and only if* $(X2)$ *is valid.*

Proposition 3 follows from the next two lemmas.

LEMMA 4. *In the class of frames* $\langle T, \prec, \mathcal{B}\rangle$ *where* $\mathcal{B}$ *is transitive, validity of* $(X2)$ *implies validity of* (X_{mem}).

PROOF. We give a syntactic proof ("PL" stands for "Propositional Logic"). Recall that in a frame $\langle T, \prec, \mathcal{B}\rangle$, $\mathcal{B}$ is transitive if and only if the axiom $B\phi \to BB\phi$ is valid in the frame (see [4]).

[8] Reflexivity requires that $x\mathcal{B}x$ for all x, while transitivity requires that $x\mathcal{B}z$ whenever $x\mathcal{B}y$ and $y\mathcal{B}z$.

1.	$PBB\phi \to BPB\phi$	(instance of $(X2)$, valid by hypothesis)
2.	$B\phi \to BB\phi$	(valid because of transitivity of $\mathcal{B}$)
3.	$PB\phi \to PBB\phi$	(2, inference rule RK$\diamond$: see [4][p. 116])
4.	$PB\phi \to BPB\phi$	(1, 3, PL). ∎

LEMMA 5. *In the class of frames $\langle T, \prec, \mathcal{B}\rangle$ where $\mathcal{B}$ is reflexive, validity of (X_{mem}) implies validity of $(X2)$.*[9]

PROOF. We give a syntactic proof.[10] Recall that in a frame $\langle T, \prec, \mathcal{B}\rangle$, $\mathcal{B}$ is reflexive if and only if the axiom $B\phi \to \phi$ is valid in the frame (see [4]).

1.	$PB\phi \to BPB\phi$	(hypothesized validity of (X_{mem}))
2.	$B\phi \to \phi$	(valid because of reflexivity of $\mathcal{B}$)
3.	$PB\phi \to P\phi$	(2, inference rule RK$\diamond$: see [4][p. 116])
4.	$BPB\phi \to BP\phi$	(3, inference rule RK: see [4][p. 114]).
5.	$PB\phi \to BP\phi$	(1, 4, PL). ∎

We now turn to a property which is related to memory of past beliefs. As part of the definition of extensive game it is usually required that if two nodes belong to the same information set of a player, then it is not the case that one node precedes the other (see, for example, [5][p. 81]). Violation of this property has been called "absent-mindedness" (see [10]). We use the expression "backward time uncertainty" to refer to a generalization of absent-mindedness to our setup.

DEFINITION 6. At $x \in T$ there is *backward time uncertainty* if there exists a $t \in T$ such that $t \prec x$ and $x\mathcal{B}t$.

The following lemma states that in frames that satisfy $(\Pi 2)$ backward time uncertainty "propagates into the past".

LEMMA 7. *Let $\langle T, \prec, \mathcal{B}\rangle$ be a frame that satisfies* $(\Pi 2)$. *Then the following is true for every $x \in T$: if at x there is backward time uncertainty, then there exists a $t \in T$ such that (1) $t \prec x$ and (2) at t there is backward time uncertainty.*

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[9] Lemma 5 can be proved under a weaker assumption than reflexivity, namely converse transitivity. Given a binary relation R on T, let $R(t) = \{x \in T : tRx\}$. While transitivity requires that if $x \in R(t)$ then $R(x) \subseteq R(t)$, *converse transitive* requires that if $R(x) \subseteq R(t)$ then $x \in R(t)$. Note that a relation which is transitive and converse transitive is not necessarily reflexive nor euclidean (a relation R is euclidean if $x \in R(t)$ implies $R(t) \subseteq R(x)$). Thus the corresponding modal operator $\Box$ does not necessarily satisfy truth ($\Box\phi \to \phi$) nor negative introspection ($\neg\Box\phi \to \Box\neg\Box\phi$). For example, let $T = \{t_1, t_2, t_3\}$ and $R = \{(t_1, t_1), (t_2, t_1), (t_2, t_3), (t_3, t_3)\}$. Then R is transitive and converse transitive but neither reflexive nor euclidean.

[10] This proof was suggested to me by Johan van Benthem.

PROOF. Let x be such that there exists a t with $t \prec x$ and $x\mathcal{B}t$. By (Π2) (letting $x' = t$) there exists a t' such that $t\mathcal{B}t'$ and $t' \prec t$. Thus at t there is backward time uncertainty. ∎

The following proposition says that a frame where the temporal precedence relation $\prec$ is *well-founded*[11] cannot have backward time uncertainty if it satisfies (Π2). For example, a (finite or infinite) rooted tree is well-founded. Thus the typical dynamic decision problem or extensive game satisfies this property.

PROPOSITION 8. *Let $\langle T, \prec, \mathcal{B}\rangle$ be a frame that satisfies* (Π2). *If $\prec$ is well-founded, then there is no $x \in T$ at which there is backward time uncertainty.*

PROOF. Suppose that there is a $t_1 \in T$ at which there is backward time uncertainty. By Lemma 7 there is an infinite sequence $\langle t_1, t_2, ...\rangle$ such that, for all $i \geq 1$, $t_{i+1} \prec t_i$ and at t_{i+1} there is backward time uncertainty, contradicting the assumption that $\prec$ is well-founded. [12] ∎

Since, when $\mathcal{B}$ is reflexive, (Π_{mem}) implies (Π2) (cf. Lemma 5), in Proposition 8 the hypothesis that the frame satisfies (Π2) can be replaced by the hypothesis that the frame satisfies (Π_{mem}) and $\mathcal{B}$ is reflexive.[13]

3. Actions

We now extend the framework by adding actions. The frames that we consider now are of the form $\langle T, \prec, \mathcal{B}, A, \{R_a\}_{a\in A}\rangle$ where A is a set of actions and, for every $a \in A$, R_a is a *partial function* on T, that is, a binary relation satisfying the property that if tR_ax and tR_ay then $x = y$. The interpretation of tR_at' is: by taking action a at state t the individual can bring about state t'. Note that it is possible that for some $t \in T$ and all $a \in A$,

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[11] That is, there is no infinite sequence $\langle t_1, t_2, ...\rangle$ such that, for all $i \geq 1$, $t_{i+1} \prec t_i$. In other words, time does not extend infinitely into the past and there are no $\prec$-cycles.

[12] If $\prec$ is not well-founded, then (Π2) is consistent with there being backward time uncertainty at every state, as the following frame shows: $T = \{t, x\}$, $\prec = \mathcal{B} = \{(t, x), (x, t)\}$. In this example $\prec$ contains a cycle, while in the following example there are no $\prec$-cycles: $T = \mathbb{Z}$ (where $\mathbb{Z}$ denotes the set of positive or negative integers), for all $m, n \in \mathbb{Z}$, $m \prec n$ if and only if $m < n$, and $\mathcal{B}$ is the universal relation on $\mathbb{Z}$, that is, $\mathcal{B} = \mathbb{Z} \times \mathbb{Z}$. This frame satisfies ($\Pi$2) and there is backward time uncertainty at every state.

[13] As pointed out above, the weaker hypothesis of converse transitivity is sufficient. When $\mathcal{B}$ is not converse transitive, (Π_{mem}) is compatible with backward time uncertainty, as the following example shows: $T = \{t', t, x\}$, $\prec = \{(t', t), (t', x), (t, x)\}$ and $\mathcal{B} = \{(x, t)\}$.

$R_a(t) \stackrel{def}{=} \{t' \in T : tR_at'\}$ is empty. In such a case the individual does not have any actions available at state t.[14]

With every relation R_a we associate two modal operators: $\Box_a$ and its inverse $\Box_a^{-1}$. The interpretation of $\Box_a\phi$ is "*after* action a it will the case that ϕ" and the interpretation of $\Box_a^{-1}\phi$ is "if action a was performed, then *before* that it was the case that ϕ ". The truth conditions are as usual:

$t \models \Box_a\phi$ if and only if $t' \models \phi$ for all t' such that tR_at', and

$t \models \Box_a^{-1}\phi$ if and only if $t'' \models \phi$ for all t'' such that $t''R_at$.

Let $\Diamond_a$ be the dual of $\Box_a$ and $\Diamond_a^{-1}$ the dual of $\Box_a^{-1}$, i.e. $\Diamond_a\phi \stackrel{def}{=} \neg\Box_a\neg\phi$ and $\Diamond_a^{-1}\phi \stackrel{def}{=} \neg\Box_a^{-1}\neg\phi$.

We make the following assumption about each relation R_a:

$$\text{if } \; tR_at' \; \text{ then } \; t \prec t'. \tag{Π_{sub}}$$

(Π_{sub}) says that each R_a is a subrelation of $\prec$, which means that *actions can only affect the future.* It is straightforward to show that (Π_{sub}) is characterized by the following axiom:

$$G\phi \to \Box_a\phi \tag{X_{sub}}$$

which states that if, at every future time, it will be the case that ϕ then after action a it will be the case that ϕ.

The same action can be available at different states and lead to different outcomes. For example, the action of opening the window may lead to a state where the floor is wet, if it is raining outside, or to a state where the floor is dry, if it is not raining outside. The individual might or might not know which outcome will occur, depending on whether she can tell what the weather is like (for example, she might be blind). Let r be the proposition "it is raining", w the proposition "the floor is wet" and let a denote the action of opening the window. Figure 6 shows three possibilities. As before, the temporal precedence relation $\prec$ is denoted by continuous arrows and the belief relation $\mathcal{B}$ by dotted arrows. We represent the relation R_a by writing the label a next to a continuous arrow. Consider state t_1. In Figure 6*a* the individual knows that it is not raining and correctly believes that if she opens the window the floor will not get wet. In Figure 6*b* she is erroneously convinced that is raining (perhaps because she hears the sound of water

[14] A transition from t to x that does not correspond to an action (i.e. $t \prec x$ but, for all $a \in A$, not tR_ax) could represent either a choice by "Nature" or an action taken by another individual. In this paper the focus is on a single individual and A is the set of actions available to that individual.

and, being blind, does not realize that it is a sprinkler) and thus incorrectly believes that if she opens the window the floor will get wet. In Figure 6*c* she is uncertain as to whether or not it is raining and, therefore, she is uncertain as to the effect of opening the window. In all three cases we have that $t_1 \models \Box_a \neg w$. In Figure 6*a*, $t_1 \models B\Box_a \neg w$ whereas in Figure 6*b*, $t_1 \models B\Box_a w$.

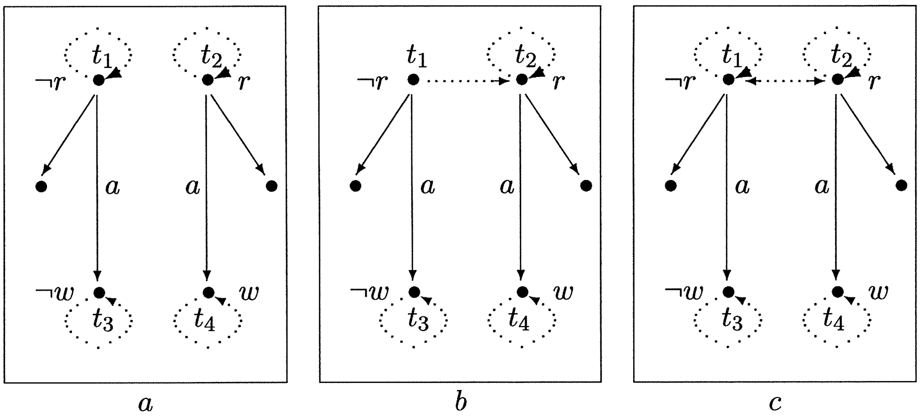

Figure 6.

The main topic that we want to address in this section is how to express the notion of remembering what one did, that is, we want to capture the fact that an individual always remembers what actions she took in the past. This is a different type of memory from the one discussed in the previous section which dealt with recollection of past beliefs. In the frame of Figure 7, for example, at state t_2 the individual remembers what she knew earlier (i.e. at state t_1) although she has forgotten what action she took at that time. Indeed, the frame of Figure 7 satisfies property (Π_{mem}) (as well as $(\Pi 2)$), although it intuitively violates memory of past actions.

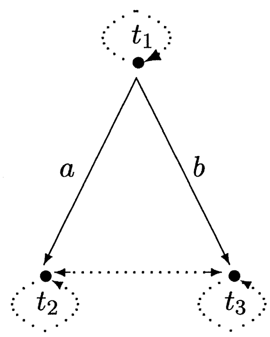

Figure 7.

In the context of extensive-form games (where the belief relation of every player is an equivalence relation) van Benthem proposes in [2] the following axiom to capture the notion of perfect recall:

$$B\Box_a\phi \to \Box_a B\phi \qquad (X_{vB})$$

which says that if the individual believes that after action a it will be the case that ϕ then after action a she will believe that ϕ.[15] In the more general framework of this paper (X_{vB}) cannot be taken as an expression of memory of past actions. Consider the frame of Figure 6*b* and state t_1. We have that $t_1 \models B\Box_a w$ and yet, since $t_3 \models B\neg w$ and $t_1 R_a t_3$, $t_1 \not\models \Box_a Bw$. Thus axiom (X_{vB}) is falsified at t_1. Indeed, whenever the individual has mistaken beliefs about the effects of a certain action and later, after taking the action, learns the truth, then axiom (X_{vB}) is necessarily violated, despite the fact that the individual might very well remember what actions she took in the past.

Consider the following property ("ar" stands for "action recall"):

If $tR_a x$ and either $x = y$ or $x \prec y$, then for every y' such that $y\mathcal{B}y'$, there exist $t', x' \in T$ such that $t'R_a x'$ and either $x' = y'$ or $x' \prec y'$. (Π_{ar})

The frames of Figure 5*a* and 5*b* (where "drink" is an action) satisfy this property, while the frame of Figure 7 violates it.

Let $\top$ be the symbol for Truth. For every action a and state x, $x \models \Diamond_a^{-1}\top$ if and only if there exists a t such that $tR_a x$. Thus the interpretation of $\Diamond_a^{-1}\top$ is "the individual has just taken action a". The following axiom says that if the individual has just taken action a, then she believes that either she has just taken action a or that some time in the past she took action a and, furthermore, she will believe this at every future time.

$$\Diamond_a^{-1}\top \;\to\; B\left(\Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top\right) \wedge GB\left(\Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top\right). \qquad (X_{ar})$$

Note that the antecedent of (X_{ar}) does *not* contain any hypothesis about what the individual believed when she acted. Thus axiom (X_{ar}) captures

[15] It should be stressed that van Benthem proposed this axiom in a different context and that transposing it to our framework does not do justice to his analysis. In particular, van Benthem referred to choices, rather than actions. The definition of choice in an extensive game (see, for example, [11]) implies that if the same choice is available at two different nodes then those two nodes must belong to the same information set of the same player (a restriction that we have not imposed on actions). Thus the notion of choice bundles together actions and epistemic states, while the purpose of our analysis is to disentangle these two notions. The analysis of this paper concerning axiom (X_{vB}) thus is not intended, nor can it be taken, to be a criticism of van Benthem's contribution.

merely memory of past actions and allows the individual to forget what she believed in the past. An example of this is the frame of Figure 5*b* where, by the following proposition, (X_{ar}) is valid, although (X_{mem}) is not.

PROPOSITION 9. *Property* (Π_{ar}) *is characterized by axiom* (X_{ar}).

PROOF. Fix a frame that satisfies property (Π_{ar}). Fix an arbitrary state x and action a and suppose that $x \models \Diamond_a^{-1}\top$. Then there exists a $t \in T$ such that $tR_a x$. First we show that $x \models B\left(\Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top\right)$. Fix an arbitrary y' such that xBy'. By (Π_{ar}) there exist $t', x' \in T$ such that $t'R_a x'$ and either $x' = y'$ or $x' \prec y'$. Thus $x' \models \Diamond_a^{-1}\top$ and $y' \models \Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top$. Hence $x \models B\left(\Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top\right)$. Next we show that $x \models GB\left(\Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top\right)$. Fix arbitrary y and y' such that $x \prec y$ and yBy'. Then by (Π_{ar}) there exist $t', x' \in T$ such that $t'R_a x'$ and either $x' = y'$ or $x' \prec y'$. Thus $x' \models \Diamond_a^{-1}\top$ and $y' \models \Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top$ and $y \models B\left(\Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top\right)$. Hence $x \models GB\left(\Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top\right)$.

To prove the converse, fix a frame that violates property (Π_{ar}). Then there exist states t, x, y, y' and an action a such that

$$tR_a x \text{ and either } x = y \text{ or } x \prec y, \tag{ix}$$

$$y\mathcal{B}y', \tag{x}$$

$$\forall t', x' \in T, \text{ if } t'R_a x' \text{ then } x' \neq y' \text{ and } x' \not\prec y'. \tag{xi}$$

By (ix),

$$x \models \Diamond_a^{-1}\top. \tag{xii}$$

By (xi), $y' \not\models \Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top$. Thus, by (x),

$$y \not\models B\left(\Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top\right). \tag{xiii}$$

If $x = y$, then by (xii) and (xiii), axiom (X_{ar}) is falsified at x. If, on the other hand, $x \prec y$, then by (xiii), $x \not\models GB\left(\Diamond_a^{-1}\top \vee P\Diamond_a^{-1}\top\right)$ and this, together with (xii), falsifies (X_{ar}) at x. ■

It is worth pointing out that property (Π_{ar}) and its characterizing axiom (X_{ar}) are rather weak. For example, they are consistent with the individual remembering what actions he took, but forgetting the order in which he took them. This is shown in the frame of Figure 8, where s is the action of shaving and d the action of getting dressed. Here the relation $\mathcal{B}$ is an equivalence relation and we have enclosed in a rounded rectangle all the states that belong to the same equivalence class. At state t the individual remembers both shaving and getting dressed, but does not remember whether he first shaved and then got dressed or the other way round. The frame of Figure 8 satisfies property (Π_{ar}) and therefore validates axiom (X_{ar}).[16]

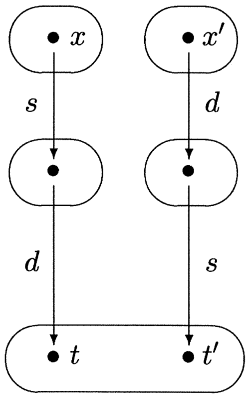

Figure 8.

We now turn to the relationship between axioms (X_{mem}) and (X_{ar}) on one hand and axiom (X_{vB}) on the other. The frame of Figure 6*b* satisfies both (Π_{mem}) and (Π_{ar}) and thus, by Propositions 2 and 9, it validates both (X_{mem}) and (X_{ar}); however, as shown above, it violates (X_{vB}). We will show that, if we replace belief with knowledge (i.e. we require the relation $\mathcal{B}$ to be an equivalence relation) and add more structure to the temporal relation $\prec$ and the action relations R_a, then the conjunction of (X_{mem}) and (X_{ar}) implies (X_{vB}). Note, however, that merely switching from beliefs to knowledge is not enough to guarantee validity of (X_{vB}) even in the presence of action recall, as shown in Figure 9*a* (where, as in Figure 8, the relation $\mathcal{B}$ is an equivalence relation and its equivalence classes are represented by rounded rectangles). Let ϕ be a formula which is true at x and y' and false at x'. Then $t \models B\Box_a\phi$ but $t \not\models \Box_a B\phi$ so that axiom (X_{vB}) is falsified at t. Note that this frame satisfies property (Π_{ar}) but violates (Π_{mem}).[17]

First we need to impose more restrictions on the relations R_a $(a \in A)$.

[16] As depicted, the relation $\prec$ is not transitive. However, validity of (X_{ar}) would not be affected if we made $\prec$ transitive by adding an arrow from x to t and an arrow from x' to t'.

[17] As depicted, the relation $\prec$ is not transitive. However, the example would not be affected if we made it transitive by adding an arrow from t' to x'. Note also that the frame of Figure 9*a* could be (part of) an extensive game where the individual under consideration moves at t, t', x and x', while a different player moves at y'. In this case, the unlabeled transition from y' to y would correspond to an action available to the other player.

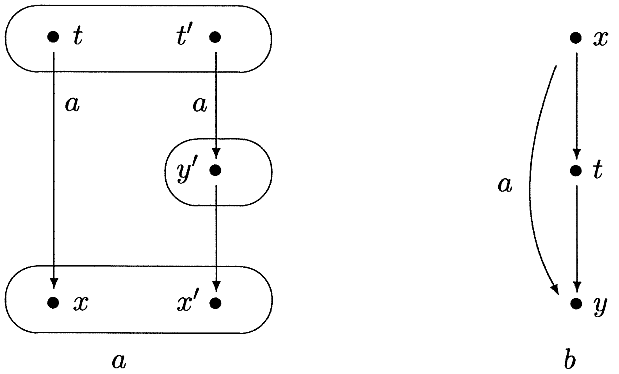

Figure 9. in part a we have $t \models \Box_a\phi \wedge B\Box_a\phi \wedge \neg\Box_a B\phi$, $t' \models \Box_a\phi$, $y' \models \phi$, $x \models \phi \wedge \neg B\phi$ and $x' \models \neg\phi$.Property $\Pi_{\min}$ rules out situations like b

The following property requires actions to correspond to minimal transitions, thereby ruling out situations like the one shown in Figure 9b.

$$\text{If } xR_ay \text{ and } t \prec y \text{ then either } t = x \text{ or } t \prec x. \qquad (\Pi_{\min})$$

PROPOSITION 10. *Property* $(\Pi_{\min})$ *is characterized by the following axiom:*
$(X_{\min})$ $\quad \Diamond_a P\phi \rightarrow \phi \vee P\phi$.

PROOF. Assume $(\Pi_{\min})$. Let $x \models \Diamond_a P\phi$. Then there exist y and t such that xR_ay, $t \prec y$ and $t \models \phi$. By $(\Pi_{\min})$ either $t = x$, in which case $x \models \phi$, or $t \prec x$, in which case $x \models P\phi$. In either case, $x \models \phi \vee P\phi$. Conversely, suppose that $(\Pi_{\min})$ is not satisfied. Then there exist states x, t and y and an action a such that xR_ay, $t \prec y$ and $t \neq x$ and $t \not\prec x$. Construct a model where, for some atomic proposition q, $||q|| = \{t\}$. Then $x \models \Diamond_a Pq$ and $x \not\models q$ and $x \not\models Pq$. Thus $(X_{\min})$ is falsified at x. ∎

The following proposition identifies a class of frames where situations like the one depicted in Figure 9a cannot arise.[18] We use the notation $t \precsim x$ to denote that either $t = x$ or $t \prec x$.

PROPOSITION 11. *Let* $\langle T, \prec, \mathcal{B}\rangle$ *be a frame where: (1) the temporal relation* $\prec$ *is transitive and well-founded, (2) the belief relation* $\mathcal{B}$ *is an equivalence relation and (3) properties* (Π_{mem}) *and* $(\Pi_{\min})$ *are satisfied. Then the following holds:*

$$\textit{if } tR_ax,\ t\mathcal{B}t',\ x\mathcal{B}x',\ t' \prec y' \textit{ and } y' \precsim x' \textit{ then } y' = x'.$$

[18] In fact, Proposition 11 rules out more than what is shown in Figure 9a, because there is no requirement that $t'R_ay'$: merely that $t' \prec y'$.

PROOF. First of all, note that the frame satisfies $(\Pi 2)$. In fact, since it satisfies (Π_{mem}), by Proposition 2 (X_{mem}) is valid in it. Hence, since $\mathcal{B}$ is reflexive, by Lemma 5 $(X2)$ is valid and, by Proposition 1, $(\Pi 2)$ is satisfied. Let t, x, x', y' and a be such that

$$tR_a x,\ t\mathcal{B}\,t',\ x\mathcal{B}\,x',\ t' \prec y' \text{ and } y' \precsim x'. \tag{xiv}$$

By Proposition 8, since $t' \prec y'$, it is not the case that $y'\mathcal{B}t'$. This implies that

$$\text{not } y'\mathcal{B}t. \tag{xv}$$

In fact, if it were the case that $y'\mathcal{B}t$, then this, together with $t\mathcal{B}\,t'$, would yield, by transitivity of $\mathcal{B}$, that $y'\mathcal{B}t'$.

Suppose (by contradiction) that $y' \prec x'$. Since $x\mathcal{B}\,x'$, by symmetry of $\mathcal{B}$ we have $x'\mathcal{B}\,x$. Thus, by $(\Pi 2)$, since $y' \prec x'$ and $x'\mathcal{B}\,x$, there exists a y'' such that

$$y'\mathcal{B}y'' \text{ and } y'' \prec x. \tag{xvi}$$

By $(\Pi_{\min})$, from $tR_a x$ and $y'' \prec x$ we get that either $t = y''$ or $y'' \prec t$. The case $t = y''$ yields a contradiction between (xv) and (xvi). Thus it must be

$$y'' \prec t. \tag{xvii}$$

By $(\Pi 2)$, from $t' \prec y'$ (by (xiv)) and $y'\mathcal{B}y''$ (by (xvi)), it follows that there exists a t'' such that

$$t'\mathcal{B}t'' \text{ and } t'' \prec y''. \tag{xviii}$$

By transitivity of $\mathcal{B}$, from $t\mathcal{B}t'$ (by (xiv)) and $t'\mathcal{B}t''$ (by (xviii)) we get that

$$t\mathcal{B}t''. \tag{xix}$$

By transitivity of $\prec$, from $t'' \prec y''$ (by (xviii)) and $y'' \prec t$ (by (xvii)) we get that $t'' \prec t$. This, in conjunction with (xix), yields backward time uncertainty at t'', contradicting Proposition 8. Since $y' \precsim x'$ and the hypothesis that $y' \prec x'$ yields a contradiction, it must be $y' = x'$. ■

We can now prove that, with additional hypotheses, memory of past beliefs and action recall imply validity of axiom (X_{vB}). In fact we will prove a stronger result. Axiom (X_{vB}) has implicit in it both memory of past actions and memory of the epistemic state the individual was in when she took the action. However, this memory requirement applies only to instants that immediately follow the action and allows the individual to forget later on what action she took and what she knew at the time. For example, the frame of Figure 10 validates axiom (X_{vB}) although at state x the individual has forgotten that she took action a and has also forgotten what she knew at the earlier instant t.[19]

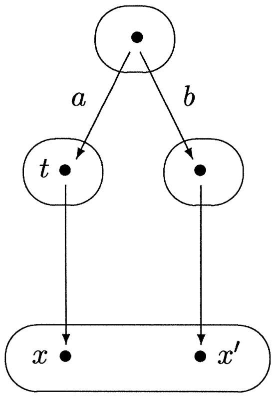

Figure 10.

The following axiom strengthens (X_{vB}) by requiring memory to be maintained at every future time, that is, by extending it into the future ("SvB" stands for "strong version of vB"):

$$B\Box_a\phi \quad\rightarrow\quad \Box_a\,(B\phi \wedge G(B\phi \vee BP\phi))\,. \qquad (X_{SvB})$$

Axiom (X_{SvB}) says that if the individual believes that after action a it will be the case that ϕ, then after action a the individual believes that ϕ *and* it will always be the case that either the individual believes that ϕ or that she believes that at some time in the past it was the case that ϕ. (X_{SvB}) is stronger than (X_{vB}): for example, in the frame of Figure 10 (X_{SvB}) is not valid, although (X_{vB}) is. On the other hand, it is clear that (X_{vB}) can be derived from (X_{SvB}).

We need to add more structure to the relations $\prec$ and R_a $(a \in A)$.

DEFINITION 12. A pair $\langle T, \prec\rangle$ is called a *branching time frame* if the temporal precedence relation $\prec$ satisfies the following properties:

(1) irreflexivity (for all $t \in T$, not $t \prec t$),

(2) transitivity (if $t \prec t'$ and $t' \prec t''$ then $t \prec t''$),

(3) backward linearity (if $t \prec y$ and $x \prec y$ then either $t = x$ or $t \prec x$ or $x \prec t$).

[19] In this frame, the only actions available to the individual are a and b; in particular, there is no action available to her at state t, so that, for every action $c \in A$ and for every formula ϕ, the formula $\Box_c\phi$ is trivially true at t.

The following property requires that if the individual has action a available then he is aware of it:

$$\text{if } tR_a x \text{ and } t\mathcal{B}t' \text{ then there exists an } x' \text{ such that } t'R_a x'. \qquad (\Pi_{aware})$$

As shown by van Benthem (2001), (Π_{aware}) is characterized by the following axiom:

$$\Diamond_a \top \to B\Diamond_a \top. \qquad (X_{aware})$$

The last property we introduce rules out situations where the same action is performed more than once along a given history:[20]

$$\text{if } \ tR_a x,\ x \precsim y \text{ and } yR_b z \ \text{ then } \ a \neq b. \qquad (\Pi_{noseq})$$

PROPOSITION 13. *Let $\langle T, \prec, \mathcal{B}\rangle$ be a frame such that $\langle T, \prec\rangle$ is a branching time frame where $\prec$ is well-founded, the belief relation $\mathcal{B}$ is an equivalence relation and property (Π_{noseq}) is satisfied. If axioms (X_{mem}), (X_{ar}) , $(X_{\min})$ and (X_{aware}) are valid in the frame then also axiom (X_{SvB}) is valid in it.*

PROOF. As a first step we prove validity of axiom (X_{vB}). Let t, a and ϕ be such that $t \models B\Box_a\phi$. We want to show that $t \models \Box_a B\phi$. Fix arbitrary x and x' such that $tR_a x$ and $x\mathcal{B}x'$. We need to show that $x' \models \phi$. By (Π_{sub}) $tR_a x$ implies $t \prec x$. Since $\mathcal{B}$ is reflexive and (X_{mem}) is valid, by Lemma 5 $(X2)$ is valid and thus, by Proposition 1, the frame satisfies $(\Pi2)$. Hence, since $t \prec x$ and $x\mathcal{B}x'$, there exists a t' such that $t\mathcal{B}t'$ and $t' \prec x'$. From $t \models B\Box_a\phi$ and $t\mathcal{B}t'$ we get that

$$t' \models \Box_a\phi. \qquad \text{(xx)}$$

By (Π_{ar}), since $tR_a x$ and $x\mathcal{B}x'$ there exist t'' and y' such that $t''R_a y'$ and

$$\text{either } \ y' = x' \ \text{ or } \ y' \prec x'. \qquad \text{(xxi)}$$

By (Π_{sub}) from $t''R_a y'$ we get $t'' \prec y'$. From this, (xxi) and transitivity of $\prec$ we get $t'' \prec x'$. Since $t' \prec x'$ and $t'' \prec x'$, by backward linearity of $\prec$ it follows that

$$\text{either } \ t' = t'' \ \text{ or } \ t' \prec t'' \ \text{ or } \ t'' \prec t'. \qquad \text{(xxii)}$$

We need to consider each of the six possible cases arising from (xxi) and (xxii).

[20] This property forces the same action performed consecutively to be identified with different action-labels, e.g. "shaving in the morning" and "shaving in the evening" or "turning left at the first junction" and "tunring left at the second junction". For instance, property (Π_{noseq}) is satisfied whenever actions are "time-stamped".

CASE 1: $y' = x'$ and $t' = t''$. In this case, since $t'' R_a y'$, we have $t' R_a x'$ and, therefore, by (xx) $x' \models \phi$.

CASE 2: $y' \prec x'$ and $t' = t''$. In this case we would have $tR_a x$, $t\mathcal{B}t'$, $x\mathcal{B}x'$, $t' R_a y'$ (and thus, by (Π_{sub}) $t' \prec y'$) and $y' \prec x'$. By Proposition 11 it follows that $y' = x'$ which is incompatible with $y' \prec x'$ and irreflexivity of $\prec$.

CASE 3: $y' = x'$ and $t' \prec t''$. In this case we would have $tR_a x$, $t\mathcal{B}t'$, $x\mathcal{B}x'$, $t' \prec t''$ and $t'' \prec x'$. By Proposition 11 it follows that $t'' = x'$ which is incompatible with $t'' \prec x'$ and irreflexivity of $\prec$.

CASE 4: $y' \prec x'$ and $t' \prec t''$. By transitivity of $\prec$, it follows from $t' \prec t''$ and $t'' \prec y'$ that $t' \prec y'$. Thus we would have $tR_a x$, $t\mathcal{B}t'$, $x\mathcal{B}x'$, $t' \prec y'$ and $y' \prec x'$. By Proposition 11 it follows that $y' = x'$ which is incompatible with $y' \prec x'$ and irreflexivity of $\prec$.

CASE 5: $y' = x'$ and $t'' \prec t'$. In this case we have $t'' \prec t'$, $t' \prec x'$ and $t'' R_a x'$. By ($\Pi_{\min}$) this requires either $t'' = t'$, which is incompatible with $t'' \prec t'$ and irreflexivity of $\prec$, or $t' \prec t''$ which, together with $t'' \prec t'$ and transitivity of $\prec$ would yield $t' \prec t'$, contradicting irreflexivity of $\prec$.

CASE 6: $y' \prec x'$ and $t'' \prec t'$. In this case since since $t' \prec x'$ and $y' \prec x'$, by backward linearity of $\prec$ either (i) $t' \prec y'$ or (ii) $t' = y'$ or (iii) $y' \prec t'$. In case (i) we have $t'' \prec t'$, $t' \prec y'$ and $t'' R_a y'$. By ($\Pi_{\min}$) this requires either $t' \prec t''$ or $t' = t''$ and in both cases we have a violation of irreflexivity of $\prec$ (as in case 5). In case (ii) we have $t'' R_a t'$. Since $tR_a x$ and $t\mathcal{B}t'$, by (Π_{aware}) there exists an x'' such that $t' R_a x''$. The conjunction of $t'' R_a t'$ and $t' R_a x''$ violates (Π_{noseq}). In case (iii) we have $t'' R_a y'$, $y' \prec t'$ and, by (Π_{aware}), $t' R_a x''$ for some x'', yielding once again a violation of (Π_{noseq}).

This completes the proof that (X_{vB}) is valid. Since the frame satisfies property ($\Pi 2$), by Proposition 1, axiom ($X2'$) is valid. The last step of the proof is a syntactic derivation of (X_{SvB}) from ($X2'$) and (X_{vB}) (as before, "PL" stands for "Propositional Logic"):

1. $B\phi \to GBP\phi$	Axiom ($X2'$)
2. $\Box_a B\phi \to \Box_a GBP\phi$	1, rule RK[21] for $\Box_a$
3. $B\Box_a\phi \to \Box_a B\phi$	Axiom (X_{vB})
4. $B\Box_a\phi \to \Box_a GBP\phi$	2, 3, PL
5. $BP\phi \to B\phi \vee BP\phi$	PL
6. $GBP\phi \to G(B\phi \vee BP\phi)$	5, rule RK for G
7. $\Box_a GBP\phi \to \Box_a G(B\phi \vee BP\phi)$	6, rule RK for $\Box_a$
8. $B\Box_a\phi \to \Box_a G(B\phi \vee BP\phi)$	4, 7, PL

9.	$B\Box_a\phi \rightarrow (\Box_a B\phi \wedge \Box_a G(B\phi \vee BP\phi))$	3, 8, PL
10.	$(\Box_a B\phi \wedge \Box_a G(B\phi \vee BP\phi)) \rightarrow$ $\Box_a(B\phi \wedge G(B\phi \vee BP\phi))$	Axiom C [22]
11.	$B\Box_a\phi \rightarrow \Box_a(B\phi \wedge G(B\phi \vee BP\phi))$	9, 10, PL. ∎

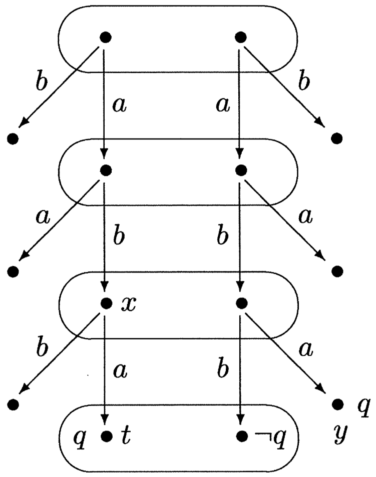

Figure 11.

It can be shown that all of the hypotheses of Proposition 13 are necessary for the result. We will only give an example to show that without property (Π_{noseq}) Proposition 13 is false. Consider the frame of Figure 11.[23] It satisfies all the hypotheses of Proposition 13 except (Π_{noseq}). Construct a model based on it, where, for some atomic proposition q, $||q|| = \{t, y\}$. Then $x \models B\Box_a q$. However, since $t \not\models Bq$, $x \not\models \Box_a Bq$, so that (X_{vB}) is falsified at at x.

4. Conclusion

Memory of past knowledge and of one's own past actions are properties normally associated with extensive-form (or dynamic) games. Indeed, most of the game theory literature restricts attention to games with perfect recall, a property introduced by Kuhn ([7]) who interpreted it as "equivalent to the

[21] See [4][p. 114].

[22] See [4][p. 114]. For any normal modal operator $\Box$, C is the axiom $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$.

[23] For simplicity we have not drawn continuous arrows that can be obtained by transitivity.

assertion that each player is allowed by the rules of the game to remember everything he knew at previous moves and all of his choices at those moves". Recently (see *Games and Economic Behavior*, 1997, Vol. 20), several contributions have focused on the dynamic inconsistency of choices and other problems that arise when the assumption of perfect recall is relaxed. Thus a general analysis of different types of memory and their interaction seems potentially useful. We have carried out such an analysis in terms of two types of memory: memory of past beliefs and memory of past actions. Our analysis covers general temporal structures (not necessarily trees or branching structures) and general epistemic states (not necessarily knowledge or even belief that satisfies the KD45 logic). The interaction of the two types of memory was also discussed as well as their relationship to an axiom proposed by van Benthem in the context of extensive games. Our analysis is both semantic and syntactic, in line with the recent literature on the logical analysis of games (see, for example, the two special issues of the *Bulletin of Economic Research*, Vol. 53, October 2001 and Vol. 54, January 2002).

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