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On the Logic of Common Belief

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Abstract. We investigate an axiomatization of the notion of common belief (knowledge) that makes use of no rules of inference (apart from Modus Ponens and Necessitation) and highlight the property of the set of accessibility relations that characterizes each axiom.

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1 Introduction

Since LEWIS's [10] and AUMANN's [1] pioneering contributions, the notions of common knowledge and common belief have been investigated thoroughly, both semantically (see, e. g. [2], [3], [5], [6], [14], [15], [16]) and syntactically (see [7], [8], [11], [12], [13]). Informally a proposition is common belief (knowledge) if everybody believes (knows) it, everybody believes (knows) that everybody believes (knows) it, and so on *ad infinitum*. From a semantic point of view there are no difficulties in capturing the informal notion, since the intersection of an infinite family of sets is a meaningful concept (semantically, the notion of common belief is captured by the transitive closure of the union of the individual accessibility relations). From a syntactic point of view, however, the informal notion cannot be captured directly because in a finitary logic formulas are required to be of finite length and, therefore, the conjunction of an infinite number of formulas is not itself a formula. Several axiomatizations of the notion of common belief (knowledge) have been offered (see [8], [11], [12] and – for a recent survey – [13]). All of them include the so called "fixed-point" axiom

$$\Box_* A \rightarrow \Box(A \wedge \Box_* A),$$

where the intended interpretation of $\Box_* A$ is "it is common belief (knowledge) that A", and that of $\Box A$ is "everybody believes (knows) that A", together with some appro-

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priate rule of inference. HALPERN and MOSES [8] use the rule

$$\frac{A \to \Box(A \wedge B)}{A \to \Box_* B},$$

while LISMONT [12] uses the rule

$$\frac{A \to \Box A}{\Box A \to \Box_* A}.$$

The purpose of this paper is to investigate an axiomatization of common belief (knowledge) that makes use of no rules of inference (apart from Modus Ponens and Necessitation) and to highlight the property of the set of accessi- bility relations that characterizes each axiom.

2 The formal system K_{n*} and its semantics

We consider a normal system with $(n+1)$ modal operators $\Box_1, \ldots, \Box_n, \Box_*$. The intended interpretation of $\Box_i A$ (for $i = 1, \ldots, n$) is "individual i believes that A" whereas $\Box_* A$ is interpreted as "it is common belief that A". The alphabet of the language consists of (1) a countable set $S = \{p_0, p_1, \ldots\}$ of *sentence letters*, (2) the *connectives* $\neg, \vee, \Box_1, \ldots, \Box_n, \Box_*$ (where $n \geq 1$ is a natural number) and (3) the *bracket symbols* (and). A *word* is a finite string of elements of the alphabet. The set $\mathcal{F}$ of *formulas* (or *sentences*) is the subset of the set of words defined recursively as follows:

(1) for every sentence letter p, $p \in \mathcal{F}$;

(2) if $A \in \mathcal{F}$, then $\neg A \in \mathcal{F}$, $\Box_* A \in \mathcal{F}$, and, for every $i = 1, \ldots, n$, $\Box_i A \in \mathcal{F}$;

(3) if $A, B \in \mathcal{F}$, then $(A \vee B) \in \mathcal{F}$.

As usual we write $(A \wedge B)$ for $\neg(\neg A \vee \neg B)$ and $(A \to B)$ for $(\neg A \vee B)$.

We denote by K_{n*} the *system* or *calculus* specified by the following axiom schemata and rules of inference:

(1) all tautologies (i. e., a suitable axiomatization of Propositional Calculus);

(2) the schema (K) (cf. CHELLAS [4])

(K) $\quad \Box_i(A \to B) \to (\Box_i A \to \Box_i B) \quad$ for every $i \in \{1, \ldots, n, *\}$;

(3) the rule of inference *Modus Ponens*:

(MP) $\quad \dfrac{A, \quad (A \to B)}{B}$;

(4) the rules of inference *Necessitation*:

(RN) $\quad \dfrac{A}{\Box_i A} \quad$ for every $i \in \{1, \ldots, n, *\}$.

We now turn to the semantics. A *standard frame* is an $(n+2)$-tuple

$$\langle W, R_1, \ldots, R_n, R_* \rangle,$$

where W is a nonempty set whose members are called *worlds* and, for $i \in \{1, \ldots, n, *\}$, R_i is a (possibly empty) binary *accessibility relation* on W. A *standard model* is an $(n+3)$-tuple

$$\mathfrak{M} = \langle W, R_1, \ldots, R_n, R_*, F \rangle,$$

where $\langle W, R_1, \ldots, R_n, R_* \rangle$ is a standard frame and $F : S \longrightarrow 2^W$ is a function from the set of sentence letters S into the set of subsets of W. We say that $\mathfrak{M}$ is *based on* the frame $\langle W, R_1, \ldots, R_n, R_* \rangle$.

Given a formula A and a standard model $\mathfrak{M} = \langle W, R_1, \ldots, R_n, R_*, F \rangle$, the *truth set* of A in $\mathfrak{M}$, denoted by $||A||^{\mathfrak{M}}$, is defined recursively as follows:

(1) If $A = p$, where p is a sentence letter, then $||A||^{\mathfrak{M}} = F(p)$;

(2) $||\neg A||^{\mathfrak{M}} = W - ||A||^{\mathfrak{M}}$ (i. e., $||\neg A||^{\mathfrak{M}}$ is the complement of $||A||^{\mathfrak{M}}$);

(3) $||A \vee B||^{\mathfrak{M}} = ||A||^{\mathfrak{M}} \cup ||B||^{\mathfrak{M}}$;

(4) for all $i \in \{1, \ldots, n, *\}$,
$$||\Box_i A||^{\mathfrak{M}} = \{\alpha \in W : \text{ for all } \beta \text{ such that } \alpha R_i \beta,\ \beta \in ||A||^{\mathfrak{M}}\}.$$

If $\alpha \in ||A||^{\mathfrak{M}}$ we say that *A is true at world α in model $\mathfrak{M}$*. An alternative notation for $\alpha \in ||A||^{\mathfrak{M}}$ is $\models_\alpha^{\mathfrak{M}} A$ and an alternative notation for $\alpha \notin ||A||^{\mathfrak{M}}$ is $\not\models_\alpha^{\mathfrak{M}} A$. A formula A is *valid in model $\mathfrak{M}$* if and only if $\models_\alpha^{\mathfrak{M}} A$ for all $\alpha \in W$.

The following proposition is a straightforward extension of a well-known result in modal logic (for a proof see HALPERN and MOSES [8]).

Proposition 1. *The system K_{n*} is sound and complete with respect to the class of standard models, that is,*

(i) *every theorem of K_{n*} is valid in every standard model;*

(ii) *if a formula A is valid in every standard model, then A is a theorem of K_{n*}.*

3 The logic of common belief

We shall consider the following axiom schemata, where $i \in \{1, \ldots, n\}$:

(S_i) $\quad \Box_* A \rightarrow \Box_i A$,

(P_i) $\quad \Box_* A \rightarrow \Box_i \Box_* A$,

(L) $\quad \Box_* (A \rightarrow \Box_1 A \wedge \cdots \wedge \Box_n A) \rightarrow (\Box_1 A \wedge \cdots \wedge \Box_n A \rightarrow \Box_* A)$.

The letter 'S' stands for 'shared belief', 'P' for 'public belief', and the letter 'L' was chosen because the corresponding axiom schema was first mentioned by LISMONT [12]. The schema (S_i) says that if it is common belief that A, then individual i believes that A; (P_i) says that if it is common belief that A, then individual i believes that it is common belief that A; finally, (L) says that if it is common belief that if A, then everybody believes that A, then if everybody believes that A, then it is common belief that A.

We say that a property $\mathcal{P}$ of the set $\{R_1, \ldots, R_n, R_*\}$ of accessibility relations *characterizes* axiom schema $\mathcal{A}$ if (1) every instance of $\mathcal{A}$ is valid in every model that satisfies $\mathcal{P}$ and (2) given a frame that does not satisfy $\mathcal{P}$, there exists a model based on that frame and an instance of $\mathcal{A}$ which is not valid in that model. In the following by 'property' we mean always 'property of the set $\{R_1, \ldots, R_n, R_*\}$ of accessibility relations'.

Proposition 2.

(i) *Axiom schema* (S_i) *is characterized by the following property:*

for all $\alpha, \beta \in W$, *if* $\alpha R_i \beta$, *then* $\alpha R_* \beta$.

(ii) *Axiom schema* (P_i) *is characterized by the following property:*[3]

for all $\alpha, \beta, \gamma \in W$, *if* $\alpha R_i \beta$ *and* $\beta R_* \gamma$, *then* $\alpha R_* \gamma$.

(iii) *Axiom schema* (L) *is characterized by the following property:*

for all $\alpha, \beta \in W$, *if* $\alpha R_* \beta$, *then there exists a sequence* $\langle \delta_1, \ldots, \delta_m \rangle$ *in* W (*with* $m \geq 2$) *and a sequence* $\langle i_1, \ldots, i_{m-1} \rangle$ *in* $\{1, \ldots, n\}$ *such that* $\delta_1 = \alpha$, $\delta_m = \beta$, *for every* $k = 2, \ldots, m$, $\alpha R_* \delta_k$, *and for every* $k = 1, \ldots, m-1$, $\delta_k R_{i_k} \delta_{k+1}$ (*that is, if* $\alpha R_* \beta$, *then there is an* R*-path from* α *to* β, *where* $R = R_1 \cup \cdots \cup R_n$, *such that for every node* γ *on this path, except possibly* α, $\alpha R_* \gamma$).

Proof. The proofs of (i) and (ii) are trivial and we omit them.

As for (iii), let $\langle W, R_1, \ldots, R_n, R_* \rangle$ be a frame that satisfies the above property. Let $\mathfrak{M}$ be a model based on it and choose an arbitrary world α in $\mathfrak{M}$ and an arbitrary formula A. Suppose that $\models_\alpha^\mathfrak{M} \Box_* (A \to \Box_1 A \wedge \cdots \wedge \Box_n A)$ and, for all $i = 1, \ldots, n$, $\models_\alpha^\mathfrak{M} \Box_i A$. We want to show that $\models_\alpha^\mathfrak{M} \Box_* A$. If there is no world which is R_*-accessible from α, then there is nothing to prove. Otherwise, let β be an arbitrary world such that $\alpha R_* \beta$. We want to show that $\models_\beta^\mathfrak{M} A$. By the assumed property, there exists a sequence $\langle \delta_1, \ldots, \delta_m \rangle$ in W and a sequence $\langle i_1, \ldots, i_{m-1} \rangle$ in $\{1, \ldots, n\}$ such that $\delta_1 = \alpha$, $\delta_m = \beta$, for every $k = 2, \ldots, m$, $\alpha R_* \delta_k$, and for every $k = 1, \ldots, m-1$, $\delta_k R_{i_k} \delta_{k+1}$. Since $\models_\alpha^\mathfrak{M} \Box_{i_1} A$ we have $\models_{\delta_2}^\mathfrak{M} A$. Since $\models_\alpha^\mathfrak{M} \Box_* (A \to \Box_1 A \wedge \cdots \wedge \Box_n A)$ and $\alpha R_* \delta_2$ we have $\models_{\delta_2}^\mathfrak{M} (A \to \Box_1 A \wedge \cdots \wedge \Box_n A)$. Thus $\models_{\delta_2}^\mathfrak{M} \Box_1 A \wedge \cdots \wedge \Box_n A$ and, therefore, $\models_{\delta_2}^\mathfrak{M} \Box_{i_2} A$. Thus $\models_{\delta_3}^\mathfrak{M} A$. Repeating this argument, we obtain $\models_{\delta_m}^\mathfrak{M} A$, i.e., $\models_\beta^\mathfrak{M} A$, as desired.

Now let $\langle W, R_1, \ldots, R_n, R_* \rangle$ be a frame where R_* does *not* satisfy the above property. Then there exist α and β such that $\alpha R_* \beta$, and either there is no R-path from α to β (recall that $R = R_1 \cup \cdots \cup R_n$) or if $\langle \delta_1, \ldots, \delta_m \rangle$ is an R-path in W with $\delta_1 = \alpha$ and $\delta_m = \beta$, then, for some $k = 2, \ldots, m$, not $\alpha R_* \delta_k$. Let

$$W_0 = \{\gamma \in W : \alpha R_* \gamma \text{ and either there is no } R\text{-path from } \alpha \text{ to } \gamma \text{ or, if there is such a path, then not } \alpha R_* \delta \text{ for some } \delta \neq \alpha \text{ on this path}\}.$$

Thus $\beta \in W_0$. Let p be a sentence letter and $\mathfrak{M}$ a model based on this frame, where $F(p) = W - W_0$.

Step 1. We show that $\models_\alpha^\mathfrak{M} \Box_* (p \to \Box_1 p \wedge \cdots \wedge \Box_n p)$. Choose an arbitrary γ such that $\alpha R_* \gamma$. We have to prove that $\models_\gamma^\mathfrak{M} (p \to \Box_1 p \wedge \cdots \wedge \Box_n p)$. Suppose that $\models_\gamma^\mathfrak{M} p$. Then $\gamma \notin W_0$. It follows that there is an R-path $\langle \delta_1, \ldots, \delta_m \rangle$ from $\alpha (= \delta_1)$ to $\gamma (= \delta_m)$ such that $\alpha R_* \delta_k$ for all $k = 2, \ldots, m$. Suppose that, for some $i = 1, \ldots, n$, $\not\models_\gamma^\mathfrak{M} \Box_i p$. Then there exists an η such that $\gamma R_i \eta$ and $\not\models_\eta^\mathfrak{M} p$. Then $\eta \in W_0$ and therefore $\alpha R_* \eta$. But then there is an R-path from α to η with $\alpha R_* \delta$ for every $\delta \neq \alpha$ on this path, implying that $\eta \notin W_0$: a contradiction.

[3] This property is a special case of a property considered by VAN DER HOEK [9, Definition 4.2(c)]. I am grateful to JOE HALPERN for this reference.

S t e p 2. We show that, for every $i = 1, \dots n$, $\models^{\mathfrak{M}}_{\alpha} \Box_i p$. Choose arbitrary $\delta \in W$ and $i \in \{1, \dots, n\}$ such that $\alpha R_i \delta$. We want to show that $\models^{\mathfrak{M}}_{\delta} p$. Suppose not. Then $\delta \in W_0$. Hence $\alpha R_* \delta$. But then, since there is an R-path from α to δ and $\alpha R_* \delta$, it follows that $\delta \notin W_0$: a contradiction.

S t e p 3. We show that $\not\models^{\mathfrak{M}}_{\alpha} \Box_* p$. This follows from the fact that $\beta \in W_0$.

Thus the formula $\Box_* (p \to \Box_1 p \wedge \dots \wedge \Box_n p) \to (\Box_1 p \wedge \dots \wedge \Box_n p \to \Box_* p)$, which is an instance of (L), is not true at α in $\mathfrak{M}$. □

R e m a r k 1. Let $R = R_1 \cup \dots \cup R_n$ and R^{Tr} be the transitive closure of R, i.e.,

> $\alpha R^{\mathrm{Tr}} \beta$ iff there exists a sequence $\langle \delta_1, \dots, \delta_m \rangle$ with $m \geq 2$ and a sequence $\langle i_1, \dots, i_{m-1} \rangle$ in $\{1, \dots, n\}$ such that $\delta_1 = \alpha$, $\delta_m = \beta$, and for every $k = 1, \dots, m-1$, $\delta_k R_{i_k} \delta_{k+1}$.

Then it is easy to see that Properties (i) and (ii) of Proposition 2 imply that $R^{\mathrm{Tr}} \subseteq R_*$. Indeed: Suppose that $\alpha R^{\mathrm{Tr}} \beta$, i.e., there are sequences $\langle \delta_1, \dots, \delta_m \rangle$ and $\langle i_1, \dots, i_{m-1} \rangle$ satisfying the above properties. We want to show that $\alpha R_* \beta$. Since $\delta_{m-1} R_{i_{m-1}} \delta_m$, by Proposition 2(i), $\delta_{m-1} R_* \delta_m$. By Proposition 2(ii), since $\delta_{m-2} R_{i_{m-2}} \delta_{m-1}$ and $\delta_{m-1} R_* \delta_m$, it follows that $\delta_{m-2} R_* \delta_m$. Repeating this argument $(m-1)$ times we obtain $\delta_1 R_* \delta_m$, i.e., $\alpha R_* \beta$, as desired. Moreover, Property (iii) of Proposition 2 implies that $R_* \subseteq R^{\mathrm{Tr}}$. Thus the conjunction of the three properties implies that $R_* = R^{\mathrm{Tr}}$ (clearly, R^{Tr} satisfies these three properties).

Let $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P} + \mathrm{L}$ be the system obtained by adding to K_{n*} the axiom schemata (L), (S_i) and (P_i) for every $i = 1, \dots, n$.

P r o p o s i t i o n 3. *The system* $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P} + \mathrm{L}$ *is sound and complete with respect to the class of models where* R_* *is the transitive closure of* $R = R_1 \cup \dots \cup R_n$.

P r o o f. Completeness: LISMONT [12] proved (soundness and) completeness for the system obtained by adding to K_{n*} the axiom schema

(F) $\quad \Box_* A \to \Box(A \wedge \Box_* A)$,

where $\Box A$ is defined as $\Box_1 A \wedge \dots \wedge \Box_n A$, and the rule of inference

(I) $$\frac{A \to \Box A}{\Box A \to \Box_* A}.$$

Now, (F) is implied by the conjunction of (S_i) and (P_i) for all $i = 1, \dots, n$, while the rule (I) is a derived rule in the system $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P} + \mathrm{L}$: Assume that $A \to \Box A$ is a theorem of $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P} + \mathrm{L}$. Then, by ($\mathrm{RN}_*$), so is $\Box_* (A \to \Box A)$. Hence, by (L) and (MP), also $\Box A \to \Box_* A$ is a theorem.

The soundness follows from Proposition 2 and Remark 1. Another way to prove soundness is to prove syntactically that (L) is a theorem of the system $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P}$ plus the inference rule (I). Such a (non-trivial) syntactical proof can be found in LISMONT [11]. □

While Proposition 3 dealt with the system K_{n*}+S+P+L, the following proposition concerns the system $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P}$.

P r o p o s i t i o n 4. *For every sequence* $\langle i_1, i_2, \dots, i_k \rangle$ *of elements of* $\{1, \dots, n\}$ *and for every formula* A, *the following is a theorem of* $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P}$:

$$\Box_* A \to \Box_{i_1} \Box_{i_2} \dots \Box_{i_k} A.$$

Proof. If $k = 1$, this is axiom (S_i). We prove that if the proposition is true for an arbitrary sequence $\langle i_2, \ldots, i_k \rangle$ with $(k-1)$ elements (with $k \geq 2$), then it is true for the sequence $\langle i_1, i_2, \ldots, i_k \rangle$ with k elements for arbitrary i_1:

1.	$\Box_* A \to \Box_{i_2} \ldots \Box_{i_k} A$	induction hypothesis
2.	$\Box_{i_1} \Box_* A \to \Box_{i_1} \Box_{i_2} \ldots \Box_{i_k} A$	1., (RN_{i_1}) – see CHELLAS [4, p. 114]
3.	$\Box_* A \to \Box_{i_1} \Box_* A$	instance of (P_{i_1})
4.	$\Box_* A \to \Box_{i_1} \Box_{i_2} \ldots \Box_{i_k} A$	2., 3., PL. $\Box$

Remark 2. Recall that, at an *informal* level, $\Box_* A$ is thought as the infinite conjunction of all formulas of the form $\Box_{i_1} \Box_{i_2} \ldots \Box_{i_k} A$, for every possible sequence $\langle i_1, i_2, \ldots, i_k \rangle$ in $\{1, \ldots, n\}$ (that is, something is commonly believed if everybody believes it, everybody believes that everybody believes it, and so on *ad infinitum*). By Proposition 4, $\Box_* A$ implies this "infinite conjunction" (that is, each element of this infinite conjunction, which is not itself a formula) in the system K_{n*}+S+P. Axiom (L) is not needed for this implication. In virtue of Proposition 3, adding axiom (L) has the effect of yielding the converse implication from the infinite conjunction to $\Box_* A$. To see that this converse implication does not hold in K_{n*}+S+P consider the following frame: $n = 2$, $W = \{\alpha, \beta\}$, $R_1 = R_2 = \emptyset$, $R_* = \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}$. This frame satisfies Properties (i) and (ii) of Proposition 2 (R_* contains the transitive closure of $R = R_1 \cup R_2$) and therefore any model based on it validates all the theorems of $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P}$. Let p be a sentence letter and let $\mathfrak{M}$ be a model where p is true at α and false at β. Then for every sequence $\langle i_1, \ldots, i_k \rangle$ in $\{1, 2\}$ the formula $\Box_{i_1} \ldots \Box_{i_k} p$ is valid in $\mathfrak{M}$. However, $\Box_* p$ is false at every world. In order for the implication from the infinite conjunction of all formulas of the form $\Box_{i_1} \ldots \Box_{i_k} A$ to $\Box_* A$ to hold, it is necessary that R_* be contained in the transitive closure of $R = R_1 \cup \cdots \cup R_n$, and this is precisely the role of axiom (L) (see Remark 1).

It is easy to check, using Proposition 2, that the axiom schemata (S), (P) and (L) form an independent set. For example, to see that (L) is not a theorem of $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P}$, consider the following frame: $n = 1$, $W = \{\alpha, \beta\}$, $R_1 = \{(\alpha, \alpha)\}$, and $R_* = \{(\alpha, \alpha), (\alpha, \beta)\}$. Note that R_* satisfies Properties (i) and (ii) of Proposition 2, hence this frame validates (S_1) and (P_1). Thus every theorem of $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P}$ is valid in every model based on this frame. If (L) were a theorem of $\mathsf{K}_{n*} + \mathrm{S} + \mathrm{P}$, then (L) would be valid in every model based on this frame, which is not the case. In fact, let p be a sentence letter and $\mathfrak{M}$ a model based on this frame where $F(p) = \{\alpha\}$. Then $\models_\alpha^{\mathfrak{M}} \Box_1 p$, and therefore $\models_\alpha^{\mathfrak{M}} (p \to \Box_1 p)$. Also, $\models_\beta^{\mathfrak{M}} (p \to \Box_1 p)$, since $\not\models_\beta^{\mathfrak{M}} p$. Thus $\models_\alpha^{\mathfrak{M}} \Box_* (p \to \Box_1 p)$. However, $\not\models_\alpha^{\mathfrak{M}} \Box_* p$. Hence, $\not\models_\alpha^{\mathfrak{M}} (\Box_1 p \to \Box_* p)$. It follows that $\not\models^{\mathfrak{M}} \Box_* (p \to \Box_1 p) \to (\Box_1 p \to \Box_* p)$.

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