A syntactic characterization of perfect recall in extensive games

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Abstract

We provide a syntactic characterization of the property of perfect recall in extensive games. The language we use is basic temporal logic with the addition of a knowledge operator for every player.

1 Introduction

In a recent, thought-provoking paper van Benthem (2001) suggests using a joint dynamic-epistemic language to analyze properties of games. The author considers games as models for this language and shows that significant conditions on games are definable by certain axioms in that language. For the class of extensive-form games, a property which is routinely

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assumed in the literature is that of perfect recall. This property was introduced by Kuhn (1953) who interpreted it as "equivalent to the assertion that each player is allowed by the rules of the game to remember everything he knew at previous moves and all of his choices at those moves". We provide a syntactic characterization of perfect recall in an extension of basic temporal logic obtained by adding a knowledge operator for every player. This is done in Section 2. In Section 3 we study an implication of perfect recall, namely the property of remembering what one knew in the past, and discuss its relationship to a similar property investigated in the computer science literature called "no forgetting". In Section 4 we relate our characterization to other results in the literature, with particular focus on the relationship between the axiom we propose for perfect recall and a simpler axiom suggested by van Benthem. Section 5 concludes.

This paper thus falls within the growing literature on the relationship between game theory and logic. The advantages of a logical analysis of game-theoretic concepts are many. First of all, the tools of modal logic have enriched the game-theoretic language by making it possible to express concepts that were previously only informally or vaguely stated. A good example is the notion of common belief in rationality and its relationship to the procedure of iterative deletion of strictly dominated strategies (see Stalnaker, 1994, and, for an overview of the role of epistemic logic in the analysis of solution concepts, Battigalli and Bonanno, 1999b). Another example is the relationship between the notion of internal consistency of a recommendation and the backward-induction algorithm (see Bonanno, 2001b). Secondly, as

Bacharach (1994, p. 21) notes,

> "Game theory is full of deep puzzles, and there is often disagreement about proposed solutions to them. The puzzlement and disagreement are neither empirical nor mathematical but, rather, concern the meanings of fundamental concepts ('solution', 'rational','complete information') and the soundness of certain arguments (that solutions must be Nash equilibria, that rational players defect in Prisoner's Dilemmas, that players should consider what would happen in eventualities which they regard as impossible). Logic appears to be an appropriate tool for game theory both because these conceptual obscurities involve notions such as reasoning, knowledge and counterfactuality which are part of the stock-in-trade of logic, and because it is a prime function of logic to establish the validity or invalidity of disputed arguments".[1]

In this vein, a logical analysis of the property of perfect recall might contribute to a better understanding of its content. Perfect recall is a central property in extensive games. Piccione and Rubinstein (1997) showed that when perfect recall is violated puzzles and paradoxes arise even in very simple decision problems.[2] There seems to be a clear need for a deeper understanding of the different aspects or components of perfect recall and their role in rational decision-making.

[1] A good example of a disputed argument in game theory is whether backward induction in perfect-information games can be derived from the hypothesis of common belief in (or knowledge of) rationality. There are those (e.g. Aumann, 1995) who claim that the answer is positive and those (e.g. Stalnaker, 1998) who claim the opposite. In a recent contribution Halpern (2001) attempts to clarify the debate by highlighting a difference in the interpretation of the counterfactuals involved in evaluating players' rationality at unreached nodes in the game tree.

[2] Their paper gave rise to an entire issue of *Games and Economic Behavior* (Vol. 20, 1997) being devoted to the consequences of relaxing perfect recall or some of its implications.

2 An axiom for perfect recall

We begin by recalling the definition of extensive form due to Kuhn (1953; see also Selten, 1975). An *extensive form* is a collection $\langle(T, \rightarrowtail, t_0), N, \{X_i\}_{i\in N}, \{\sim_i\}_{i\in N}, C\rangle$, where

- $(T, \rightarrowtail, t_0)$ is a (finite or infinite) *rooted tree* with t_0 as root. For any two nodes $t, x \in T$, $t \rightarrowtail x$ denotes that t is the *immediate predecessor* of x (or x is an *immediate successor* of t). We assume that every node has finite outdegree, that is, a finite number of immediate successors. Let $\prec$ denote the transitive closure of $\rightarrowtail$. Thus $t \prec x$ means that t is a *predecessor* of x (that is, there is a path from t to x) and we write $t \precsim x$ as a short-hand for $t = x$ or $t \prec x$. Let Z be the (possibly empty) set of *terminal nodes*, that is, nodes that have no successors and $X = T \backslash Z$ the set of *decision nodes.*

- $N = \{0, 1, ..., n\}$ is the set of *players.* Players $i = 1, ..., n$ are called *personal players*, while player 0 is called *Nature* and represents events that are not the outcome of actions taken by personal players.

- $\{X_0, X_1, ..., X_n\}$ is a partition of the set of decision nodes X. For every player $i \in N$, X_i is the set of decision nodes of player i.

- For every player $i \in N$, $\sim_i$ is an equivalence relation (that is, a binary relation which is reflexive, symmetric and transitive) on X_i satisfying the following condition: if $t, t' \in X_i$ and $t \sim_i t'$ then the number of immediate successors of t is equal to the number of immediate successors of t'. The interpretation of $t \sim_i t'$ is that player i

cannot distinguish between t and t', that is, as far as she knows, she could be making a decision either at node t or at node t'. The equivalence classes of $\sim_i$ partition X_i and are called the *information sets of player i*.[3]

- C is the *choice partition*, which, for every information set, partitions the edges out of nodes in that information set (that is, the set of ordered pairs (t, x) such that $t \rightarrowtail x$) into choices at that information set. If (t, x) belongs to choice c we write $t \rightarrowtail_c x$. The choice partition satisfies the following constraints: (1) if $t \rightarrowtail_c x$ and $t \rightarrowtail_c x'$ then $x = x'$, and (2) if $t \rightarrowtail_c x$ and $t \sim_i t'$ for some $i \in N$, then there exists an x' such that $t' \rightarrowtail_c x'$. The first condition says that a choice at a node selects a unique immediate successor, while the second condition says that if a choice is available at one node of an information set then it is available at every node in that information set.

An example of an extensive form is given in Figure 1. Here the set of players is $N = \{1, 2\}$, the set of terminal nodes is $Z = \{z_1, ..., z_6\}$ and the set of decision nodes is $X = \{t_0, t, t', x, x'\}$. The set of player 1's decision nodes is $X_1 = \{t_0, t\}$, while the set of player 2's decision nodes is $X_2 = \{t', x, x'\}$. The equivalence relations are $\sim_1 = \{(t_0, t_0), (t, t)\}$ and $\sim_2 = \{(t', t'),\ (x, x), (x, x'), (x', x), (x', x')\}$. Thus, for example, player 2's information sets are $\{t'\}$ and $\{x, x'\}$. We use the graphic convention of representing an information set as a rounded rectangle enclosing the corresponding nodes, if there are at least two nodes, while

[3] The following additional constraint, usually imposed on Nature, plays no role in our analysis: if $t \sim_0 t'$ then $t = t'$.

if an information set is a singleton we do not draw anything around it. Furthermore, since all the nodes in an information set belong to the same player, we write the corresponding player only once inside the rectangle. The choices are shown by labeling the corresponding edges in such a way that two edges belong to the same choice if and only if they are assigned the same label. Thus, for example, $x \rightarrowtail_g z_2$ and $x' \rightarrowtail_g z_4$ so that player 2's choice g is $\{(x, z_2), (x', z_4)\}$. As a further example of our notation, we have that $t \rightarrowtail x$, that is, x is an immediate successor of t, and $t \prec z_3$, that is, z_3 is a successor of t.

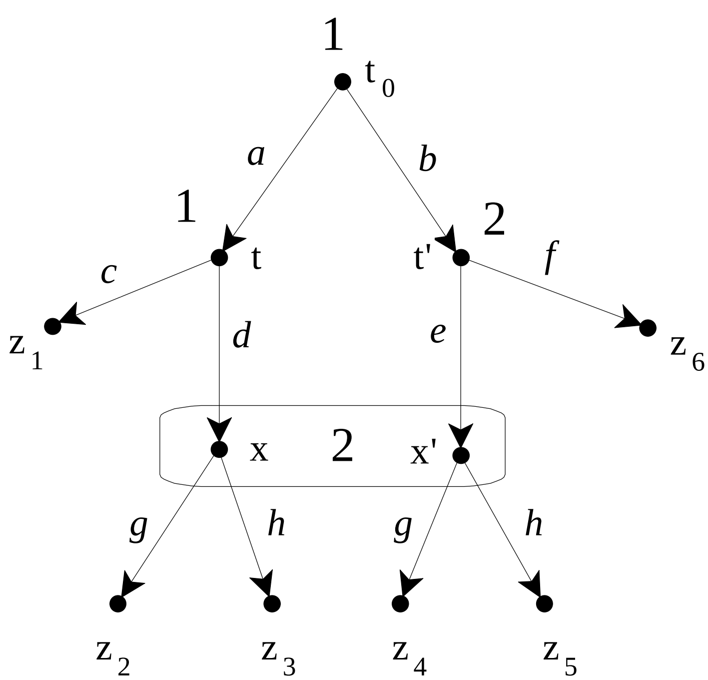

Figure 1
Example of extensive form

Traditionally, game theorists have restricted attention to games with perfect recall. This property, which was introduced by Kuhn (1953), requires that if a node y of player i comes after a choice c at a previous node t of player i himself, then every node in the information set that contains y also comes after the same choice c at the information set that contains t. Formally,[4]

$$\text{For every player } i \in N \text{, for all nodes } t, y, y' \in X_i \text{ and } x \in T \text{ and for every choice } c \text{, if } t \rightarrowtail_c x,\ x \precsim y \text{ and } y \sim_i y' \text{ then there exist nodes } t' \in X_i \text{ and } x' \in T \text{ such that } t \sim_i t',\ t' \rightarrowtail_c x' \text{ and } x' \precsim y'. \qquad (PR)$$

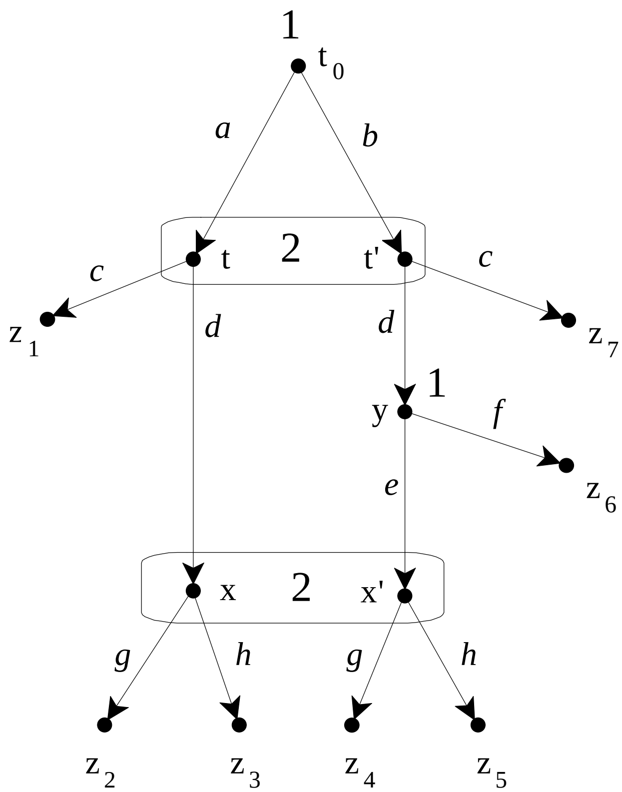

Figure 2
An extensive form with perfect recall

The extensive form of Figure 1 violates this property, since x' is a decision node of player 2 which comes after choice e at the earlier decision node t' of player 2, x belongs to the same

[4] The following definition is Selten's (1975) reformulation of Kuhn's original property which was stated in terms of pure strategies.

information set of player 2 as x' and yet x does not come after choice e. On the other hand, the extensive form of Figure 2 does satisfy perfect recall.

We now provide an axiomatic characterization of perfect recall using temporal epistemic logic. We interpret the precedence relation $\prec$ as a temporal relation and associate with it the standard future and past operators from basic temporal logic, denoted by G and H (see, for example, Burgess, 1984, or Goldblatt, 1992). Furthermore, to the equivalence relation $\sim_i$ of player i we associate a knowledge operator $\mathcal{K}_i$, and to every choice c a modal operator $\Box_c$. The intended interpretation is as follows:

$G\phi$: "it is **G**oing to be the case at every future time that ϕ"

$H\phi$: "it **H**as always been the case that ϕ"

$K_i\phi$: "player i **K**nows that ϕ"

$\Box_c\phi$: "after choice c it will be the case that ϕ".

Furthermore, for every player i we add a modal constant (or nullary modality) $turn_i$, whose intended interpretation is "it is player i's turn to move".

The formal language is built in the usual way from a countable set S of atomic propositions, the connectives $\neg$ (for "not") and $\vee$ (for "or") and the modal operators.[5] Let $F\phi \stackrel{def}{=} \neg G\neg\phi$ and $P\phi \stackrel{def}{=} \neg H\neg\phi$. Thus the interpretation is:

[5]Thus the set Φ of formulas is defined inductively as follows: (1) $turn_i \in \Phi$, (2) $q \in \Phi$ for every atomic proposition $q \in S$, (3) if $\phi, \psi \in \Phi$ then all of the following belong to Φ: $\neg\phi$, $\phi \vee \psi$, $G\phi$, $H\phi$ and $K_i\phi$. See, for example, Chellas (1984). The connectives $\wedge$ (for "and") and $\rightarrow$ (for "if ... then") are defined as usual: $\phi \wedge \psi \stackrel{def}{=} \neg(\neg\phi \vee \neg\psi)$ and $\phi \rightarrow \psi \stackrel{def}{=} \neg\phi \vee \psi$.

$F\phi$: "at *some* **F**uture time it will be the case that ϕ"

$P\phi$: "at *some* **P**ast time it was the case that ϕ".

Given an extensive form one obtains a *model based on it* by adding a function $V : S \to 2^T$ (where 2^T denotes the set of subsets of the set of nodes T) that associates with every atomic proposition $q \in S$ the set of nodes at which q is true. Truth of a formula ϕ at a node t, denoted by $t \models \phi$, is defined inductively as follows:

if q is an atomic proposition, $t \models q$ if and only if $t \in V(q)$,

$t \models \neg\phi$ if and only if $t \nvDash \phi$ and $t \models \phi \vee \psi$ if and only if either $t \models \phi$ or $t \models \psi$ (or both),

$t \models G\phi$ if and only if $t' \models \phi$ for all t' such that $t \prec t'$,

$t \models H\phi$ if and only if $t'' \models \phi$ for all t'' such that $t'' \prec t$,

$t \models K_i\phi$ if and only if $t' \models \phi$ for all t' such that $t \sim_i t'$,

$t \models \Box_c\phi$ if and only if $t \to_c x$ implies $x \models \phi$,

$t \models turn_i$ if and only if $t \in X_i$.

Thus $G\phi$ ($H\phi$) is true at node t if and only if ϕ is true at *every* successor (predecessor) of t, while $F\phi$ ($P\phi$) is true at t if and only if ϕ is true at *some* successor (predecessor) of t. Furthermore, $K_i\phi$ is true at node t if and only if either $t \notin X_i$ (see Remark 1 below) or $t \in X_i$ *and* ϕ is true at every node in the information set of player i containing t. Finally, $\Box_c\phi$ is true at t if and only if ϕ is true at the immediate successor of t following choice c and

$turn_i$ is true at node t if and only if t is a decision node of player i.

We denote by $||\phi||$ the truth set of formula ϕ, that is, $||\phi|| = \{t \in T : t \models \phi\}$. A formula ϕ is *valid in a model* if $t \models \phi$ for all $t \in T$, that is, if ϕ is true at every node. A formula ϕ is *valid in an extensive form* if it is valid in every model based on it.

We say that a property of extensive forms is *characterized by* an axiom if the axiom is valid in every extensive form that satisfies the property and, conversely, if whenever the axiom is valid in an extensive form then the extensive form satisfies the property. The "axiomatizations" provided in this paper are characterizations of properties of extensive games by means of axioms. This is known in modal logic as 'frame definability (or distinguishability)' (see, for example, Blackburn *et al.*, 2001, p. 125).

Consider the following axiom:

$$(turn_i \wedge K_i \Box_c \phi) \rightarrow \Box_c \left(K_i \left(\phi \vee P\phi \right) \wedge GK_i \left(\phi \vee P\phi \right) \right). \qquad (A_{PR})$$

Axiom (A_{PR}) says that, if it is player i's turn to move and she knows that after choice c it will be the case that ϕ, then after choice c player i knows that either ϕ is true now or was true in the past and, furthermore, she will always know this.

Before we prove the characterization result we draw attention to the following fact, which is well-known in modal logic (see Chellas, 1984, p. 77).

Remark 1 *Fix a player i and a node t that does* not *belong to player i, that is $t \notin X_i$. Then, for every formula ϕ, the formula $K_i\phi$ is true at t, that is, $t \models K_i\phi$. In fact, for $t \not\models K_i\phi$ to be* *the case there would have to exist a t' such that $t \sim_i t'$ and $t' \nvDash \phi$. But $\sim_i$ is defined only on X_i (that is, $\sim_i$ is empty-valued on $T \backslash X_i$). Similarly, if c is* not *a choice at t, then $t \models \Box_c \phi$ for every formula ϕ.*

Proposition 2 *The property of perfect recall (PR) is characterized by axiom (A_{PR}), that is,*

(A) the axiom is valid in every extensive form with perfect recall, and

(B) if the axiom is valid in an extensive form, then the extensive form satisfies perfect recall.

Proof. (A) Fix an extensive form that satisfies perfect recall and any model based on it. Fix an arbitrary player i, node t, choice c and formula ϕ and suppose that $t \models turn_i \wedge K_i \Box_c \phi$. Then

$$t \in X_i \text{ and, for every } t' \text{ and } x' \text{ such that } t \sim_i t' \text{ and } t' \rightarrowtail_c x', \;\; x' \models \phi. \tag{1}$$

If c is not a choice at t, then $t \models \Box_c \psi$ for every formula ψ (see Remark 1), thus, in particular, for $\psi = K_i(\phi \vee P\phi) \wedge GK_i(\phi \vee P\phi)$. Suppose, therefore, that c is a choice at t and let x be such that $t \rightarrowtail_c x$. We want to show that $x \models K_i(\phi \vee P\phi) \wedge GK_i(\phi \vee P\phi)$. If $x \notin X_i$ then $x \models K_i \psi$ for every formula ψ (see Remark 1). Suppose, therefore, that $x \in X_i$. Fix an arbitrary y' such that $x \sim_i y'$. By perfect recall there exist t' and x' such that $t \sim_i t'$, $t' \rightarrowtail_c x'$ and $x' \precsim y'$. If $x' = y'$ then by (1) $y' \models \phi$. If $x' \prec y'$ then by (1) $y' \models P\phi$. Thus in either case $y' \models \phi \vee P\phi$ and therefore $x \models K_i(\phi \vee P\phi)$. Next we show that $x \models GK_i(\phi \vee P\phi)$. Fix an arbitrary y such that $x \prec y$. If $y \notin X_i$ then $y \models K_i(\phi \vee P\phi)$ trivially (see Remark 1). Suppose, therefore, that $y \in X_i$. Fix an arbitrary y' such that $y \sim_i y'$. By perfect recall there exist t' and x' such that $t \sim_i t'$, $t' \rightarrowtail_c x'$ and $x' \precsim y'$. By (1) $x' \models \phi$ and thus $y' \models \phi \vee P\phi$. Hence $y \models K_i(\phi \vee P\phi)$ and $x \models GK_i(\phi \vee P\phi)$.

Next we prove the contrapositive of (B), namely that if an extensive form does *not* satisfy perfect recall then axiom (A_{PR}) is not valid in it. Fix an extensive form that violates perfect recall. Then there exist a player i, nodes $t, y, y' \in X_i$ and $x \in T$, and a choice c at t such that $t \rightarrowtail_c x$, $x \precsim y$ and $y \sim_i y'$ and

$$\text{for all } t' \text{ and } x' \text{ if } t \sim_i t' \text{ and } t' \rightarrowtail_c x' \text{ then } x' \neq y' \text{ and } x' \nprec y'. \tag{2}$$

Construct a model based on this extensive form where, for some atomic proposition q, the truth set of q is the set of immediate successors of nodes in the information set of player i that contains t following choice c, that is,

$$||q|| = \{x' \in T : t' \rightarrowtail_c x' \ for\ some\ t'\ such\ that\ t \sim_i t'\}.$$

Then

$$t \models turn_i \wedge K_i \Box_c q. \tag{3}$$

By (2) $y' \nvDash q \vee Pq$. Hence, since $y \sim_i y'$, $y \nvDash K_i (q \vee Pq)$. Thus, if $x = y$ then $x \nvDash K_i (q \vee Pq)$, while if $x \prec y$ then $x \nvDash GK_i (q \vee Pq)$. In either case $x \nvDash K_i (q \vee Pq) \wedge GK_i (q \vee Pq)$. Thus, since $t \rightarrowtail_c x$, $t \nvDash \Box_c (K_i (q \vee Pq) \wedge GK_i (q \vee Pq))$. This, together with (3), falsifies axiom (A_{PR}) at node t. ■

3 Perfect recall, memory and extended partitions

An implication of perfect recall is that at any decision node of player i the player remembers what she knew at earlier decision nodes of hers. Formally, property (PR) implies the following property ('KM' stands for 'Knowledge Memory')

$$\begin{array}{l} \text{If } t, y \in X_i \text{ and } t \prec y, \text{ then for every } y' \text{ such that } y \sim_i y' \\ \text{there exists a } t' \in X_i \text{ such that } t \sim_i t' \text{ and } t' \prec y'. \end{array} \tag{KM}$$

(KM) says that if t and y are decision nodes of player i and t precedes y, then every node y' in the information set of player i that contains y has a predecessor in the information set that contains t.[6]

[6] Perfect recall is a strengthening of this property in that it requires, in addition, that all the nodes in the information set containing y be preceded by the same *choice* at the information set containing t.

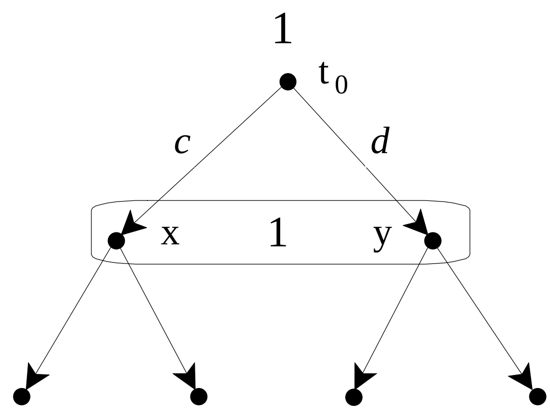

Figure 3
An extensive form that satisfies (KM) but not (PR)

This property was first discussed in game theory by Okada (1987, p. 89).[7] An essentially identical property, called 'no forgetting', was discussed by Halpern and Vardi (1986, p. 313).[8] Every extensive form with perfect recall clearly satisfies this property. However, there are extensive forms that violate perfect recall and yet satisfy property (KM). An example is the extensive form of Figure 3.

Within the context of systems of runs, where the knowledge of each agent is specified at every instant (unlike extensive forms, where a player's knowledge is specified only at those nodes where the player has to move) Halpern and Vardi (1986) propose the following axiom to capture the notion of 'no forgetting': $K_i\hat{G}\phi \rightarrow \hat{G}K_i\phi$,where $\hat{G}\phi$ stands for $(\phi \wedge G\phi)$, that is, 'ϕ is true now and at every future time'.[9] We shall consider the following, essentially identical, version of this axiom: $K_iG\phi \rightarrow GK_i\phi$. Adapted to the context of extensive forms,

[7] See also Kline (2000, p. 288), who refers to it as 'occurrence memory' and Ritzberger (1999, p. 77), who calls it 'strong ordering'.

[8] It was later renamed 'perfect recall' in Fagin *et al.* (1995, p. 129). However it it not the same property as perfect recall as defined above for extensive games.

[9] The authors use the symbol $\Box$ instead of $\hat{G}$.

where the knowledge of a player is specified only when it is her turn to move, the appropriate version of this axiom is the following:

$$turn_i \wedge K_i G\phi \to GK_i\phi. \qquad (HV)$$

This axiom says that if it is player i's turn to move and he knows that it will always be the case that ϕ, then at every future time player i will know that ϕ. To the best of our knowledge, the following characterization has not been proved in the literature.

Proposition 3 *Property (KM) is characterized by axiom (HV), that is,*

(A) the axiom is valid in every extensive form that satisfies (KM), and

(B) if the axiom is valid in an extensive form, then the extensive form satisfies property (KM).

Proof. Fix an extensive form that satisfies property (KM) and any model based on it. Let t be a node such that, for some formula ϕ and player i, $t \models turn_i \wedge K_i G\phi$. Then $t \in X_i$. Fix an arbitrary y such that $t \prec y$. We need to show that $y \models K_i\phi$. If $y \notin X_i$ then $y \models K_i\phi$ trivially (cf. Remark 1). Suppose, therefore, that $y \in X_i$. Fix an arbitrary y' such that $y \sim_i y'$. By property (KM), there exists a t' such that $t \sim_i t'$ and $t' \prec y'$. Since $t \models K_i G\phi$ and $t \sim_i t'$, $t' \models G\phi$. Thus, since $t' \prec y'$, $y' \models \phi$. Hence $y \models K_i\phi$.

Conversely, fix an extensive game that violates property (KM). Then there exist a player i, nodes $t, y \in X_i$ with $t \prec y$, and a node y' such that $y \sim_i y'$ and

$$\text{for all } t' \text{ such that } t \sim_i t',\ t' \not\prec y'. \qquad (4)$$

Let q be an atomic proposition and construct a model where the truth set of q is the set of successors of nodes in the information set of player i that contains t, that is, $||q|| = \{x \in T : t' \prec x \ \ for \ \ some \ \ t' \ \ with \ \ t \sim_i t'\}$. Then $t \models turn_i \wedge K_i Gq$. By (4), $y' \nvDash q$. Hence, since $y \sim_i y'$, $y \nvDash K_i q$. Thus, since $t \prec y$, $t \nvDash GK_i q$ and axiom (HV) is falsified at t. ■

As noted above, in the computer science and logic literature that deals with the interaction of knowledge and time, the knowledge of an agent is specified at every instant. One could do the same in extensive forms by extending the information partition of player i from the set X_i of his decision nodes, to the set of all nodes T. That is, one could define, for every player i, an equivalence relation $R_i \subseteq T \times T$ satisfying the property that if $x \in X_i$ then, for all $y \in T$, xR_iy if and only if $x \sim_i y$. In other words, the restriction of R_i to X_i coincides with $\sim_i$, so that the original information sets are preserved. Having done so, one can then define an extension of property (KM) to the entire set T, that is, dropping the restriction that $t, y \in X_i$:

$$\text{If } t \prec y \text{ and } yR_iy', \text{ then there exists a } t' \text{ such that } tR_it' \text{ and } t' \prec y'. \qquad (KM_{EXT})$$

With a proof similar to that used for Proposition 3 it can be shown that (KM_{EXT}) is characterized by the following simplified version of axiom (HV), obtained by dropping $turn_i$ from the antecedent:

$$K_iG\phi \to GK_i\phi. \qquad (HV_{EXT})$$

The following proposition shows that (HV_{EXT}) is equivalent to the following axioms, which capture more explicitly the notion of remembering what one knew in the past:

$$PK_i\phi \to K_iP\phi \qquad (MEM_K)$$

$$K_i\phi \to GK_iP\phi. \qquad (GMEM_K)$$

Axiom (MEM_K) (which stands for 'memory of past knowledge') says that if at some point in the past player i knew that ϕ then she knows now that some time in the past it was the case that ϕ. While this axiom is backward-looking, $(GMEM_K)$ is forward-looking: it says that if player i knows that ϕ now then at every future time she will know that some time in the past it was the case that ϕ.

Proposition 4 *The three axioms (HV_{EXT}), (MEM_K) and $(GMEM_K)$ are equivalent.*

Proof. We give a syntactic proof. First we prove the equivalence of (HV_{EXT}) and $(GMEM_K)$.

Derivation of (HV_{EXT}) from $(GMEM_K)$ (PL stands for 'Propositional Logic'):

1.	$K_iG\phi \to GK_iPG\phi$	instance of $(GMEM_K)$
2.	$PG\phi \to \phi$	axiom of basic temporal logic (see Burgess, 1984, p. 93)
3.	$K_iPG\phi \to K_i\phi$	2, normality of K_i
4.	$GK_iPG\phi \to GK_i\phi$	3, normality of G
5.	$K_iG\phi \to GK_i\phi$	1, 4, PL.

Derivation of $(GMEM_K)$ from (HV_{EXT}):

1.	$K_iGP\phi \to GK_iP\phi$	instance of (HV_{EXT})
2.	$\phi \to GP\phi$	axiom of basic temporal logic (see Burgess, 1984, p. 93)
3.	$K_i\phi \to K_iGP\phi$	2, normality of K_i
4.	$K_i\phi \to GK_iP\phi$	1, 3, PL.

Next we prove the equivalence of $(GMEM_K)$ and (MEM_K).

Derivation of $(GMEM_K)$ from (MEM_K):

1.	$PK_i\phi \to K_iP\phi$	axiom (MEM_K)
2.	$GPK_i\phi \to GK_iP\phi$	1, normality of G
3.	$\psi \to GP\psi$	axiom of basic temporal logic (see Burgess, 1984, p. 93)
4.	$K_i\phi \to GPK_i\phi$	instance of 3
5.	$K_i\phi \to GK_iP\phi$	2, 4, PL.

Derivation of (MEM_K) from $(GMEM_K)$:

1.	$K_i\phi \to GK_iP\phi$	axiom $(GMEM_K)$
2.	$F\neg K_iP\phi \to \neg K_i\phi$	1, PL (recall that $G = \neg F \neg$)
3.	$HF\neg K_iP\phi \to H\neg K_i\phi$	2, normality of H
4.	$PK_i\phi \to PGK_iP\phi$	3, PL (recall that $G = \neg F \neg$ and $H = \neg P \neg$)
5.	$PG\psi \to \psi$	axiom of basic temporal logic (see Burgess, 1984, p. 93)
6.	$PGK_iP\phi \to K_iP\phi$	instance of 5
7.	$PK_i\phi \to K_iP\phi$	4, 6, PL. ■

An interesting question is whether, given an extensive form that satisfies property (KM) (or the stronger property of perfect recall), it is possible to define an extension $R_i \subseteq T \times T$ of $\sim_i$ that satisfies (KM_{EXT}). In other words, given an extensive form where, *whenever it is his turn to move*, a player remembers what he knew at earlier decision nodes of his, is it possible to define an extension of the information partition (from the set X_i of player i's decision nodes to the set T of all nodes) that satisfies the property that at *every* instant the player remembers what he knew in the past? It turns out that there are cases where it cannot be done. Consider, for example, the extensive form of Figure 2, which satisfies perfect recall and therefore also property (KM). The set of decision nodes of player 2 is $X_2 = \{t, t', x, x'\}$ and her information sets are $\{t, t'\}$ and $\{x, x'\}$. There are only two possible extensions of player 2's information partition to the set of all nodes: one where nodes t_0 and y belong to two separate equivalence classes, namely $\{t_0\}$ and $\{y\}$, and the other where they belong to the same equivalence class, namely $\{t_0, y\}$.[10] In other words, the possible extensions are:

$R_2^A = \{(t_0,t_0),(t,t),(t,t'),(t',t),(t',t'),(y,y),(x,x),(x,x'),(x',x),(x',x')\}$, and

[10] For simplicity we have ignored terminal nodes. Considering also terminal nodes, there are more – although less natural – possibilities (e.g. including node t_0 in the same equivalence class as a terminal node, thereby postulating that at node t_0 player 2 is uncertain as to whether the game has not started yet or has already been completed). It is straightforward to verify that all these possibilities involve violations of property (KM_{EXT}).

$R_2^B = \{(t_0,t_0),(t_0,y),(y,t_0),(y,y),(t,t),(t,t'),(t',t),(t',t'),(x,x),(x,x'),(x',x),(x',x')\}$.

Both R_2^A and R_2^B violate property (KM_{EXT}): R_2^A because $y \prec x'$, $x'R_2^A x$ and there is no node v such that $yR_2^A v$ and $v \prec x$; R_2^B because $t' \prec y$, $yR_2^B t_0$ and there is no node v such that $t'R_2^B v$ and $v \prec t_0$. Intuitively, with R_2^A player 2 learns at node y that player 1 has to move (and choose between e and f), but later (at node x') forgets this piece of information. With R_2^B, on the other hand, at node y player 2 forgets what he knew previously (at node t'), namely that the game had started and player 1 had made a choice between a and b.

There is, however, a class of games for which the extension discussed above is possible: it is the class of von Neumann games. For every $t \in T$, we denote by $\ell(t)$ the number of predecessors of t (i.e. the length of the path from the root to t). Thus $\ell(t_0) = 0$ and if $x \rightarrowtail y$ then $\ell(y) = \ell(x) + 1$. The following definition is taken from Kuhn (1953; p. 52 of Kuhn, 1997).

Definition 5 *An extensive form is* von Neumann *if, whenever t and x are decision nodes of player i that belong to the same information set of player i, the number of predecessors of t is equal to the number of predecessors of x. Formally:* $\forall i \in N$, $\forall t, x \in X_i$, *if* $t \sim_i x$ *then* $\ell(t) = \ell(x)$.

For example, the extensive forms of Figures 1 and 3 are von Neumann, while that of Figure 2 is not. Battigalli and Bonanno (1999a) show that given a von Neumann extensive form that satisfies perfect recall, for every player i there is an extension of the information partition of player i from X_i to the entire set of nodes T that satisfies property (KM_{EXT}). A simple adaptation of their proof shows that the hypothesis of perfect recall can be weakened to the hypothesis that the extensive form satisfies property (KM). On the other hand, Bonanno (2001a) shows that if an extensive form is such that, for every player i, there is an extension of the information partition from X_i to the entire set of nodes T that satisfies property (KM_{EXT}), then the extensive form is von Neumann. Thus we have the following proposition.

Proposition 6 *Fix an arbitrary extensive form that satisfies property* (KM). *Then there exists, for every player* i, *an extension of the information partition from* X_i *to* T *that satisfies* (KM_{EXT}) *if and only if the extensive form is von Neumann.*

4 Related literature

Several semantic characterizations of perfect recall have been proposed in game theory. Okada (1987, Proposition 4, p. 89) shows that, within the class of extensive forms that satisfy property (KM), perfect recall is equivalent to complete inflation, a property introduced by Dalkey (1953).[11] Ritzberger (1999, Theorem 2, p. 81) offers a number of alternative semantic characterizations of perfect recall, involving properties such as complete inflation, weak and strong recall, etc. Neither of these authors offers an axiomatization or syntactic

[11] An extensive form is *completely inflated* if there is no information set that contains an isolated subset. A subset v of an information set h of player i is isolated in h if, for every two nodes $y \in v$ and $y' \in h \backslash v$, there exists another information set h' of player i and two distinct choices c and c' at h' such that y comes after choice c and y' comes after choice c'.

characterization of perfect recall or its component properties, such as (KM).

The interaction of knowledge and time has been studied extensively in computer science. In particular, as noted above, property (KM) is essentially identical to a property introduced by Ladner and Reif (1986) and Halpern and Vardi (1986). In the latter it was called 'no forgetting'. Halpern and Vardi (1986) also provide a sound and complete axiomatization of systems that satisfy 'no forgetting' and are synchronous (i.e. the agents have access to an external clock). The key axiom is $K_i \bigcirc \phi \to \bigcirc K_i \phi$, where $\bigcirc$ is the 'next time' operator, that is, $t \models \bigcirc \phi$ if and only if ϕ is true at every immediate successor of t. A thorough account of sound and complete axiomatizations of systems where knowledge and time interact is given in Halpern *et al.* (2002).

A paper which is closely related to ours is van Benthem (2001), which views extensive games as models for a joint dynamic-epistemic language. Properties of those games are shown to be definable by certain axioms in that combined language. In particular, van Benthem proposes the following axiom to capture perfect recall:

$$(turn_i \wedge K_i \Box_c \phi) \to \Box_c K_i \phi. \qquad (vB)$$

Axiom (vB) says that if it is player i's turn to move and he knows that after choice c it will be the case that ϕ then after choice c player i knows that ϕ. The following results show that van Benthem implicitly restricted attention to von Neumann games.

Proposition 7 *Axiom (vB) is valid in every von Neumann extensive form that satisfies* *perfect recall.*

Proof. Fix a von Neumann extensive form with perfect recall. Fix an arbitrary node t such that, for some player i and choice c, $t \models turn_i \wedge K_i \Box_c \phi$. Since $t \models turn_i$, $t \in X_i$, that is, t is a decision node of player i. Let y be the immediate successor of t following choice c, that is, $t \rightarrowtail_c y$. If $y \notin X_i$, then $y \models K_i \phi$ trivially. Suppose, therefore, that $y \in X_i$. Fix an arbitrary y' such that $y \sim_i y'$. By perfect recall, there exist t' and x' such that $t \sim_i t'$, $t' \rightarrowtail_c x'$ and $x' \precsim y'$. Thus, since $t \models K_i \Box_c \phi$, $x' \models \phi$. Since the game is von Neumann and $t \sim_i t'$, $\ell(t) = \ell(t')$ (recall that $\ell(x)$ denotes the number of predecessors of x). Similarly, it follows from $y \sim_i y'$ that $\ell(y) = \ell(y')$. Since y is an immediate successor of t, $\ell(y) = \ell(t) + 1$. Thus $\ell(y') = \ell(t') + 1$, that is, $y' = x'$. Hence $y' \models \phi$ and $y \models K_i \phi$ and $t \models \Box_c K_i \phi$. ■

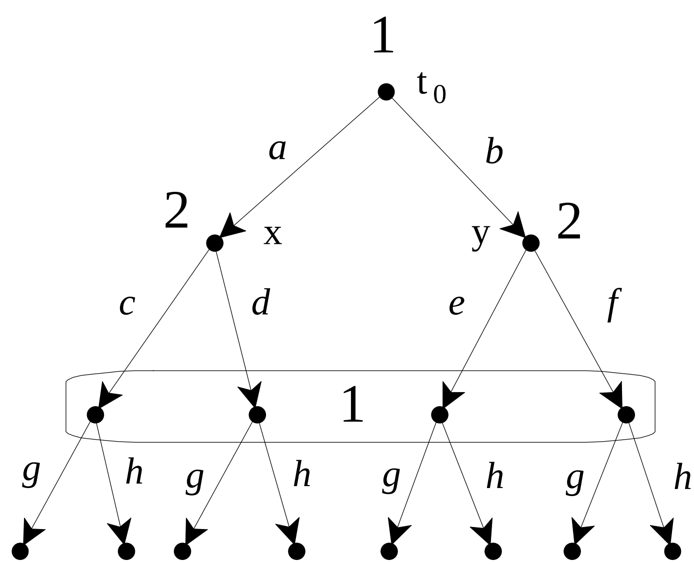

Figure 4
A von Neumann extensive form without perfect recall

The converse of Proposition 7 does not hold, that is, a von Neumann extensive form that validates axiom (vB) does not necessarily satisfy perfect recall. To see this, consider the von Neumann extensive form of Figure 4. Here axiom (vB) is trivially valid, since the immediate successors of a decision node of player i are decision nodes of the other player and thus at those nodes $K_i \phi$ is trivially true for every formula ϕ. However, the extensive form clearly violates perfect recall (indeed it even violates (KM)).

It is worth emphasizing that validity of (vB) in the extensive form of Figure 4 can still be guaranteed even if the information partition of player i were to be extended to the set of all nodes. In other words, it does not hinge on the fact that at a node that does not belong to X_i player i trivially knows everything. To see this, extend the information partition of player i in such a way that the equivalence class of a node not in X_i consists of that node only. For example, for player 1 we would add the following 'extended information sets': $\{x\}$ and $\{y\}$. It is easy to verify that with such an extended partition axiom (vB) remains valid. However, the following result says that if, for every player i, there is an extended partition that satisfies (KM_{EXT}) (implying that the extensive form is von Neumann) then validity of (vB) in the *extended* structure does guarantee that the extensive form satisfies perfect recall.

Proposition 8 *Let G be an extensive form and, for every player $i \in N$, let $R_i \subseteq T \times T$ be an extension of the information partition $\sim_i$ from X_i to T that satisfies property (KM_{EXT}) (thus, by Proposition 6, G is von Neumann). If axiom (vB) is valid in $\langle G, \{R_i\}_{i \in N} \rangle$ then G satisfies perfect recall.*

Proof. Let G be an extensive form and R_i an extension of $\sim_i$ that satisfies property (KM_{EXT}). Suppose that axiom (vB) is valid in $\langle G, \{R_i\}_{i \in N} \rangle$ but G does not satisfy perfect recall. Then there exist a player i, nodes $t, y, y' \in X_i$ and $x \in T$, and a choice c at t such that $t \rightarrowtail_c x$, $x \precsim y$, $y \sim_i y'$ and

$$\text{for all } t' \text{ and } x' \text{ if } t \sim_i t' \text{ and } t' \rightarrowtail_c x' \text{ then } x' \neq y' \text{ and } x' \not\prec y'. \tag{5}$$

Construct a model where, for some atomic proposition q, the truth set of q is the set of immediate successors of nodes in the information set containing t following choice c, that is, $||q|| = \{v \in T : t' \rightarrowtail_c v \text{ for some } t' \text{ such that } t \sim_i t'\}$. Then $t \models turn_i \wedge K_i \Box_c q$. By the postulated validity of axiom (vB), $t \models \Box_c K_i q$. Thus, since $t \rightarrowtail_c x$, $x \models K_i q$. Hence, for every x' such that xR_ix', $x' \models q$ and therefore, by construction of $||q||$, $t' \rightarrowtail_c x'$ for some t' such that $t \sim_i t'$. Since $x \precsim y$ and $y \sim_i y'$, by (KM_{EXT}), there exists a x' such that xR_ix' and $x' \precsim y'$, contradicting (5). ■

The following characterization result is a consequence of Propositions 6, 7 and 8. It should be noted that van Benthem (2001) explicitly dealt with extended structures where the knowledge of each player is specified at every node.

Corollary 9 *Let G be an extensive form and, for every player $i \in N$, let $R_i \subseteq T \times T$ be an extension of the information partition $\sim_i$ from X_i to T that satisfies property (KM_{EXT}). Then G is von Neumann and, furthermore, G satisfies perfect recall if and only if axiom (vB) is valid in $\langle G, \{R_i\}_{i \in N} \rangle$.*

By Proposition 7, axiom (vB) is valid in von Neumann extensive forms that satisfy perfect recall. However, in extensive forms that are not von Neumann it is no longer true that perfect recall guarantees validity of axiom (vB). To see this, consider the extensive form of Figure 2, which satisfies perfect recall. Let q be an atomic proposition and construct a model where the truth set of q is $\{x, y\}$. The model is shown in Figure 5, where only the relevant portion of the tree is highlighted and the formulas that are true at a node are written next to it. Since $||q|| = \{x, y\}$, $\Box_d q$ is true at both t and t' and therefore at t player 2 knows that $\Box_d q$: $t \models K_2 \Box_d q$. Furthermore, since $t \in X_2$, $t \models turn_2$. Thus $t \models turn_2 \wedge K_2 \Box_d q$. Since q is false at x' and $x \sim_2 x'$, it is not the case that player 2 knows that q at x: $x \not\models K_2 q$. Thus, since $t \rightarrow_d x$, $t \not\models \Box_d K_2 q$. Thus we have constructed a model where axiom (vB) is falsified at node t. Note that this example does not hinge on the fact that player 2 trivially knows everything at node y (since it is not a decision node of his: see Remark 1). In fact, we could take an arbitrary extension R_2 of player 2's information partition $\sim_i$. Whatever the equivalence class of R_2 that contains node y (one possibility is $\{y\}$), the example is not affected. Of course, as explained in the discussion of Figure 2, such an extension R_2 cannot satisfy (KM_{EXT}).

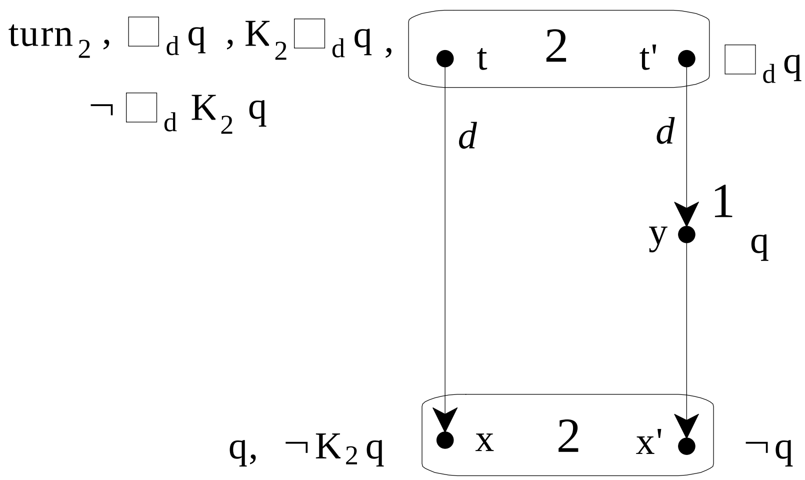

Figure 5
A portion of the extensive form of Figure 2

While axiom (A_{PR}) characterizes perfect recall in the class of all extensive forms, axiom (vB) provides a characterization only in the subclass of von Neumann extensive forms that satisfy the conditions of Corollary 9. The appeal of axiom (vB) is in the fact that it is based on a simple commutation of two modal operators: in the antecedent we have $K_i\Box_c$ and in the consequent $\Box_c K_i$. It was pointed out to me by Johan van Benthem that with an opportune choice of temporal operators also axiom (A_{PR}) can be seen as involving a commutation of operators. To show this we define two new temporal operators, $\hat{G}$ and $\hat{H}$, as follows:

$$\hat{G}\phi \stackrel{def}{=} \phi \wedge G\phi \text{ and } \hat{H}\phi \stackrel{def}{=} \phi \wedge H\phi.$$

Thus $\hat{G}\phi$ says that ϕ is true now and at every future time and $\hat{H}\phi$ says that ϕ is true now and was true at every past instant. Let $\hat{P}$ be the dual of $\hat{H}$, that is, $\hat{P}\phi \stackrel{def}{=} \neg\hat{H}\neg\phi$. Then $\hat{P}\phi$ is equal to $(\phi \vee P\phi)$. Using these operators, axiom (A_{PR}) can be rewritten as

$$turn_i \wedge K_i\Box_c\phi \rightarrow \Box_c\hat{G}K_i\hat{P}\phi. \qquad (A_{PR})$$

The following proposition and proof were suggested to me by Johan van Benthem.

Proposition 10 *Axiom (A_{PR}) is equivalent to the following*

$$turn_i \wedge K_i\Box_c\hat{G}\phi \rightarrow \Box_c\hat{G}K_i\phi. \qquad (vBA_{PR})$$

Proof. Derivation of (vBA_{PR}) from (A_{PR}):

1.	$turn_i \wedge K_i\Box_c\hat{G}\phi \rightarrow \Box_c\hat{G}K_i\hat{P}\hat{G}\phi$	instance of (A_{PR})
2.	$\hat{P}\hat{G}\phi \rightarrow \phi$	theorem of basic temporal logic
3.	$K_i\hat{P}\hat{G}\phi \rightarrow K_i\phi$	2, normality of K_i
4.	$\hat{G}K_i\hat{P}\hat{G}\phi \rightarrow \hat{G}K_i\phi$	3, normality of $\hat{G}_i$
5.	$\Box_c\hat{G}K_i\hat{P}\hat{G}\phi \rightarrow \Box_c\hat{G}K_i\phi$	4, normality of $\Box_c$
6.	$turn_i \wedge K_i\Box_c\hat{G}\phi \rightarrow \Box_c\hat{G}K_i\phi$	1, 5, PL.

Derivation of (A_{PR}) from (vBA_{PR}):

1.	$turn_i \wedge K_i\Box_c\hat{G}\phi \rightarrow \Box_c\hat{G}K_i\phi$	axiom (vBA_{PR})
2.	$\phi \rightarrow \hat{P}\phi$	theorem of propositional logic
3.	$K_i\phi \rightarrow K_i\hat{P}\phi$	2, normality of K_i
4.	$\hat{G}K_i\phi \rightarrow \hat{G}K_i\hat{P}\phi$	3, normality of $\hat{G}$
5.	$\Box_c\hat{G}K_i\phi \rightarrow \Box_c\hat{G}K_i\hat{P}\phi$	4, normality of $\Box_c$
6.	$turn_i \wedge K_i\Box_c\hat{G}\phi \rightarrow \Box_c\hat{G}K_i\hat{P}\phi$	1, 5, PL. ■

The appeal of axiom (vBA_{PR}) is that it involves a commutation of the operators K_i and the concatenation of $\Box_c$ and $\hat{G}$, thus yielding an axiom which is close in spirit to axiom (vB).

5 Conclusion

Perfect recall is a central property in extensive games. Kuhn (1953) introduced this property and showed that the equivalence between mixed strategies and behavioral strategies holds only in games with perfect recall.[12]The purpose of this paper was to contribute to a better understanding of perfect recall by providing a syntactic characterization of it in a temporal-epistemic logic and by relating it to similar properties studied in computer science and logic. The recent debate on the paradoxes of decision-making when perfect recall is lacking (see *Games and Economic Behavior,* Vol. 20, 1997) points to the need for a deeper understanding of the different aspects or components of perfect recall and their role in rational decision-making. For example, Kline (2002) finds that a weaker condition than perfect recall is both necessary and sufficient for the equivalence between *ex ante* optimality and time consistency.[13] Bonanno (2003) studies different aspects of memory that are implied by the notion of perfect recall and their relationship. The role of perfect recall, or memory in general, in rational decision-making is in need of further exploration.

[12]A *pure* strategy of player i in an extensive game is a function that associates with every information set of player i a choice at that information set. A *mixed* strategy of player i is a probability distribution over the set of pure strategies. A *behavioral* strategy of player i is a collection of probability distributions, one for each information set of player i. Each probability distribution is over the set of choices at the corresponding information set.

[13]A strategy of a player is *ex ante* optimal if it maximizes the player's expected payoff before the actual play of the game. It is time consistent if the player does not wish to modify it during the play of the game.

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