Reprinted from JOURNAL OF ECONOMIC THEORY

Vol. 35, No. 2, April 1985
Printed in Belgium

Limited Knowledge of Demand and Oligopoly Equilibria

GIACOMO BONANNO

London School of Economics, Houghton Street, London WC2A 2AE, England

AND

E. CHRISTOPHER ZEEMAN

Mathematics Institute, University of Warwick, Coventry CV4 7AL, England

Received October 2, 1984

In a standard model of oligopoly with differentiated products, the existence of an equilibrium at which the first-order conditions for profit maximisation are simultaneously satisfied for all firms is proved and this is done without imposing any restrictions on the demand functions. This is an equilibrium in the following sense: although some firms may not necessarily be maximising their profits, nevertheless if each firm's knowledge of demand is limited to the linear approximation of its own demand curve, then it will believe that it is indeed maximising its profits. *Journal of Economic Literature* Classification Number: 022. © 1985 Academic Press, Inc.

1. INTRODUCTION

Most of the theorems which establish the existence of a Nash equilibrium in models of oligopoly and monopolistic competition make use of the assumption that each firm's profit function is quasi-concave in the firm's decision variable. Roberts and Sonnenschein [3] showed that this assumption implies restrictions on the shape of the demand curves which are arbitrary, since they cannot be derived from standard conditions on consumers' preferences.

In this paper we make one step in the direction of dealing with the problem pointed out by Roberts and Sonnenschein. In a standard model of oligopoly (or monopolistic competition) with differentiated products we show that there exists a price vector at which the first-order conditions for profit maximisation are simultaneously satisfied for all firms. We prove this result *without imposing any restrictions at all on the shape of the demand curves*. We call such a price vector an equilibrium. The use of the word

0022-0531/85 $3.00

"equilibrium" is justified if firms' knowledge of demand is very limited. By this we mean that at any given *status quo* each firm knows only the linear approximation of its own demand curve at that point and believes it to be the demand curve it faces. If this is the case, then at an equilibrium each firm will believe that it is maximising its profits, even though it may not be at a global maximum of its "true" profit function (and, indeed, it may even be at a local minimum).

This notion of equilibrium was first studied by Silvestre [4] within a general equilibrium model. The explanation given by Silvestre [4, p. 436] is that "firms may perform price experiments in a neighbourhood of a given *status quo* and such experiments linearly approximate the variation of demand." It seems, in fact, reasonable to assume that firms do not know the demand curve they face and, therefore, they can only learn about their demand functions through price experiments. Moreover, firms may limit themselves to small price changes, fearing that large increases in prices may induce customers to switch to other brands. Therefore, firms will engage in local price experiments and, by extrapolating the information collected, will formulate some conjectures about the demand curves they face. The simplest of all possible conjectures is that of a linear demand curve and we consider the case in which these conjectures are "locally correct," in the sense that the conjectural demand curve coincides with the linear approximation to the "true" demand curve.

In Section 2 we prove the existence theorem within a general game-theoretic framework and in Section 3 we apply it to a standard oligopoly (or monopolistic competition) model with differentiated products. The result we prove is that an equilibrium exists always, no matter what the shape of the demand curve is, that is, *no conditions need to be imposed on the first and second derivatives of the demand functions* (in his model Silvestre [4] required the derivative of the demand function to be negative everywhere and bounded away from zero).[1]

Since the problem pointed out by Roberts and Sonnenschein is present both at the partial and at the general equilibrium level, we have chosen a partial equilibrium framework in order to make the analysis as simple as possible.

[1] This result was first proved by Bonanno [1] for a generic set of duopoly models under the assumption of zero costs of production. The expression "infinitesimal Nash equilibrium" was used there. Bonanno also considers the existence of "local Nash equilibria" characterized by the fact that firms know their "true" demand curves but only in a neighbourhood of any given *status quo*. At a local Nash equilibrium, therefore, unlike at an infinitesimal Nash equilibrium, firms are at a local maximum of their "true" profit functions (even though they may not be maximising profits).

2. The Existence Theorem for n-Person Games

We shall consider the class of n-person non-cooperative games in which each player's strategy set is the compact interval $[-1, 1]$. Let x_i be a strategy for player i ($i = 1, ..., n$) and let C be the n-dimensional cube

$$C = \{x \in \mathbb{R}^n / -1 \leqslant x_i \leqslant 1, i = 1, ..., n\}.$$

We shall assume that the payoff function of player i, $f_i: C \to \mathbb{R}$, is continuous and satisfies the following hypotheses:

(1) $\partial f_i/\partial x_i: C \to \mathbb{R}$ exists and is continuous.

(2) Boundary conditions:

$$\begin{aligned} \partial f_i/\partial x_i &> 0 \quad \text{if} \quad x_i = -1, \\ &< 0 \quad \text{if} \quad x_i = 1. \end{aligned}$$

We can regard f_i as a function of x_i parametrised by $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n$ and for each value of the parameters the function looks like Fig. 1.

Note that we do not require f_i to be differentiable with respect to the parameters (i.e. f_i need not be differentiable with respect to x_j for $j \neq i$).

Definition 1. An *equilibrium* is a point $x \in C$ such that

$$(\partial f_i/\partial x_i)(x) = 0 \qquad \text{for all } i = 1, ..., n.$$

We shall justify the use of the word "equilibrium" in the next section.

Theorem 1. *There exists an equilibrium in the interior of C.*

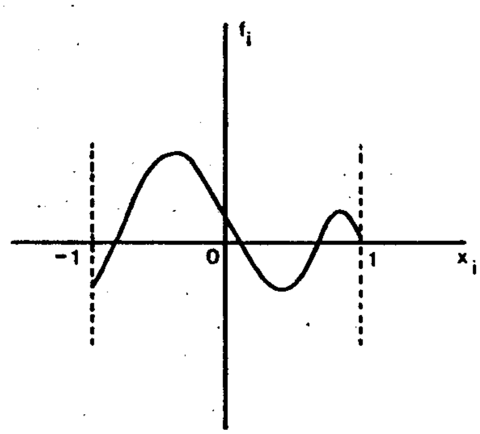

Fig. 1. The payoff function of player i, f_i, for given values of the parameters $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n$.

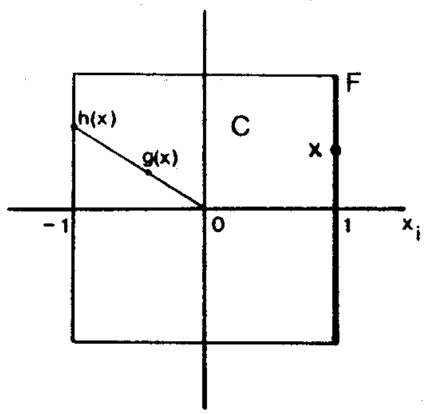

FIG. 2. The function $h(x)$.

Proof. Suppose not. Let ∂C denote the boundary of C. Let $g: C \to \mathbf{R}^n$ be given by $g = (g_1, ..., g_n)$, $g_i = \partial f_i / \partial x_i$. Then g is continuous by hypothesis (1) and $0 \notin g(C)$ by our supposition. Therefore, for each x in C, $g(x) \neq 0$ and so the ray from the origin through $g(x)$ is well defined, and consequently meets ∂C in a unique point, which we call $h(x)$ (see Fig. 2). By construction $h: C \to \partial C$ is continuous and therefore by the Brouwer fixed point theorem has a fixed point $x = h(x)$. Since the image of h is a subset of ∂C, this fixed point must belong to ∂C. Therefore it belongs to some face F of C, given by $x_i = \pm 1$, for some i. If $x_i = 1$, then $g_i(x) < 0$ by hypothesis (2) and therefore also $h_i(x) < 0$, which implies that $h(x) \neq x$. Similarly, if $x_i = -1$, then $g_i(x) > 0$ and therefore $h_i(x) > 0$, which implies that $h(x) \neq x$ (see Fig. 2). In either case we have a contradiction. Hence the theorem is true.

3. Application to Oligopoly and Monopolistic Competition

As an application of the result of the previous section we consider the case of n firms (indexed by $i = 1, ..., n$) producing differentiated products (which are close substitutes) and using prices as their strategic variables. Each firm therefore faces its own demand curve, which depends also on the prices charged by the other firms. We can interpret a firm's demand function as a parametrised family of demand functions, the parameters being the prices of the other firms. Our result applies both to the case of oligopoly and to the case of monopolistic competition, that is, it is irrelevant whether a change in the price of firm i has a large or small effect on the demand curve faced by firm j, for $j \neq i$.

In order to simplify the analysis we shall follow Silvestre [4] and assume that all firms have a constant-return-to-scale technology. The cost function of firm i is therefore given by

$$C_i(q_i) = c_i q_i,$$

where $q_i \geqslant 0$ is the output of firm i, and c_i is a positive constant. Let I_i be the interval $[c_i, \infty)$ and let $I = I_1 \times \cdots \times I_n$. Since no firm will choose a price $p_i < c_i$, in considering the demand function of firm i we can restrict our attention to the set I. Let therefore $D_i: I \to \mathbb{R}$ be the demand function of firm i. We shall use the notation $I_{-i} = \prod_{j \neq i} I_j$ and denote an element of I_{-i} by p_{-i}. Hence $p \in I$ can be written as $p = (p_i, p_{-i})$. We shall make the following assumptions about the demand functions (see Fig. 3a).

Assumption 1 (A1). D_i is continuous on I. Furthermore, for each $p_{-i} \in I_{-i}$ there exists a p_i' such that:

(a) $c_i < p_i' < \infty$;

(b) $D_i(p_i, p_{-i}) > 0$ for each $p_i \in [c_i, p_i')$;

(c) $D_i(p_i, p_{-i}) = 0$ for each $p_i \in [p_i', \infty)$;

(d) p_i' varies continuously with $p_{-i} \in I_{-i}$ and we shall therefore denote it by $p_i'(p_{-i})$.

(A1) says that each firm faces a finite reservation price for its product, which depends continuously on the prices charged by the other firms, and that for prices below that reservation price demand is positive. Assumption (a) that $p_i'(p_{-i}) > c_i$ for all $p_{-i} \in I_{-i}$ is not a strong one, since, by definition of I_{-i}, each component p_j of p_{-i} is greater than or equal to c_j. It can, therefore, be interpreted as saying that when all firms sell at zero profits demand for each product is positive.

If a firm increases its price we expect the other firms to face a higher demand and therefore a higher reservation price. However, we shall assume that reservation prices are bounded away from infinity (this assumption is

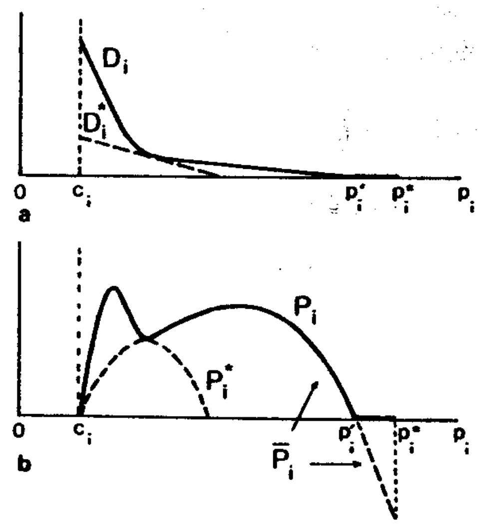

FIG. 3. (a) The demand function D_i and the conjectural demand function D_i^*. (b) The profit function P_i and the functions P_i^* and $\bar{P}_i$.

certainly reasonable in a partial equilibrium setting and is standard, cf. Friedman [2, p. 502]).

ASSUMPTION 2 (A2). For each i there exists a $p_i^* < \infty$ such that $p_i'(p_{-i}) \leqslant p_i^*$, for all $p_{-i} \in I_{-i}$.

Finally, we shall assume that the demand curve is continuously differentiable in the firm's price and has a negative slope at the point at which it intersects the price axis.

ASSUMPTION 3 (A3). For each i and for each $p_{-i} \in I_{-i}$ the derivative $\partial D_i/\partial p_i$ exists and is continuous on the interval $[c_i, p_i'(p_{-i})]$. Furthermore, for each $p_{-i} \in I_{-i}$

$$\frac{\partial D_i}{\partial p_i}(p_i'(p_{-i}), p_{-i}) < 0$$

(this should be interpreted as left-hand derivative, since by (A1) the right-hand derivative must be zero).

(A3) is weaker than the usual assumption (cf. Friedman [2, p. 501]) that D_i is twice continuously differentiable with respect to all its arguments. Note that we do not require D_i to be differentiable with respect to p_j for $j \neq i$. Note also that *we do not require the demand function to be downward-sloping.*

Figure 3a shows the graph of D_i for a given vector of prices charged by the other firms (note that, contrary to common use, we measure price on the horizontal axis). It can be seen from Fig. 3a that we do not impose any restrictions at all on the shape of the demand function (and indeed, as remarked above, the demand curve could even be upward-sloping for some values of p_i).

The profit function of firm i, $P_i: I \to \mathbb{R}$, is given by

$$P_i(p) = (p_i - c_i)\, D_i(p).$$

Figure 3b shows the profit function corresponding to the demand function drawn in Fig. 3a.

In the present context the definition of equilibrium given above (Definition 1) can be restated as follows.

DEFINITION 2. An *equilibrium* is a vector of prices $p^0 \in I$ such that

$$P_i(p^0) > 0 \qquad \text{and} \qquad \frac{\partial P_i}{\partial p_i}(p^0) = 0 \qquad \text{for all } i = 1, \ldots, n.$$

As said in the Introduction, a possible justification for the use of the word "equilibrium" is the following. Firms have a very limited knowledge of their demand functions. At any given *status quo* p^0, firms only know the demand they face, $D_i(p^0)$, and the slope of the demand curve at that point, $(\partial D_i/\partial p_i)(p^0)$. Firm i conjectures that the demand function it faces, given the prices charged by the other firms, p^0_{-i}, is

$$D_i^*(p_i, p^0_{-i}) = D_i(p^0) + (p_i - p_i^0)\frac{\partial D_i}{\partial p_i}(p^0),$$

and as a consequence it believes its profit function to be

$$P_i^*(p_i, p^0_{-i}) = (p_i - c_i)\, D_i^*(p_i, p^0_{-i}).$$

D_i^* and P_i^* are shown as dashed curves in Figure 3.[2] It can be checked that if $(\partial P_i/\partial p_i)(p^0) = 0$ then $(\partial^2 P_i^*/\partial p_i^2)(p^0) < 0$ and therefore firm i believes that it is maximising its profits, given the actions taken by the other firms, even though it may not be at a global maximum of its "true" profit function.[3] Note the interesting fact that the linear approximation to the demand function at a minimum of the profit function can convert that minimum into a maximum. Figure 3 illustrates this fact: at an equilibrium a firm may even be at a local minimum of its "true" profit function and nevertheless believe that it is maximising profits.

We can now prove the following proposition.

PROPOSITION 2. *There exists an equilibrium at which each firm makes positive profits.*

Proof. Proposition 2 is merely an application of Theorem 1. We only have to show that the hypotheses of Theorem 1 can be satisfied. First of all, firm i will never choose a price $p_i > p_i^*$ (where p_i^* is as defined in (A2)). We can therefore restrict our attention to the compact interval $A_i = [c_i, p_i^*]$ and let $A = A_1 \times \cdots \times A_n$. From now on we shall consider the restriction of P_i to A. Let $\bar{P}_i\colon A \to \mathbb{R}$ be defined as follows:

$$\bar{P}_i(p) = P_i(p) \qquad \text{for} \quad p_i \in [c_i, p_i'(p_{-i})],$$

$$= (p_i - p_i'(p_{-i}))\frac{\partial P_i}{\partial p_i}(p_i'(p_{-i}), p_{-i}) \qquad \text{for} \quad p_i \in [p_i'(p_{-i}), p_i^*].$$

[2] The position of D_i^* corresponding to a critical point of P_i is a tangent whose point of contact lies midway between its intersections with the price axis and the line $p_i = c_i$. Note also that P_i^* is a parabola with axis vertical and touching P_i at its vertex.

[3] Note that even if the demand curve is not downward-sloping, nevertheless a singularity of the profit function occurs at a point where the demand curve has a negative slope, that is, $(\partial P_i/\partial p_i)(p) = 0$ implies $(\partial D_i/\partial p_i)(p) < 0$. The demand curve D_i shown in Fig. 3a consists of two straight lines smoothed at their intersection and hence P_i, shown in Fig. 3b, consists of two parabolas smoothed at their intersection.

In other words, $\bar{P}_i$ is obtained from P_i by prolonging the tangent at p_i' instead of going along the price axis. $\bar{P}_i$ is illustrated in Fig. 3b. Let $T: C \to A$ be the affine map from the n-cube C onto A defined by $T = (T_1, ..., T_n)$, $T_j(x) = (\frac{1}{2})(p_j^* + c_j + (p_j^* - c_j)\, x_j)$, $j = 1, ..., n$. Finally, define

$$f_i(x) = \bar{P}_i(T(x)).$$

By (A3) f_i satisfies hpothesis (1) of Theorem 1 and also the second boundary condition. By (A1b) $(\partial P_i/\partial p_i)(c_i, p_{-i}) > 0$ for all p_{-i} and therefore f_i satisfies also the first boundary condition. Since f_i satisfies the hypotheses of Theorem 1 and the singularities of f_i in C are mapped by T onto the singularities of $\bar{P}_i$ in A, which are singularities of P_i, Proposition 2 follows from Theorem 1.

References

1. G. Bonanno, Existence of local and infinitesimal Nash equilibria in two-person games with applications to duopoly, Working Paper N. 1, Associazione Borsisti Luciano Jona, Turin, December 1983.
2. J. Friedman, Oligopoly theory, *in* "Handbook of Mathematical Economics," Vol. II (K. J. Arrow and M. D. Intrilligator, Eds.), North-Holland, Amsterdam, 1982.
3. D. J. Roberts and H. Sonnenschein, On the foundations of the theory of monopolistic competition, *Econometrica* **45** (1977), 101–113.
4. J. Silvestre, A model of general equilibrium with monopolistic behaviour, *J. Econ. Theory* **16** (1977), 425–442.

Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium