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Temporal Interaction of Information and Belief

Abstract. The temporal updating of an agent's beliefs in response to a flow of information is modeled in a simple modal logic that, for every date t, contains a normal belief operator B_t and a non-normal information operator I_t which is analogous to the 'only knowing' operator discussed in the computer science literature. Soundness and completeness of the logic are proved and the relationship between the proposed logic, the AGM theory of belief revision and the notion of plausibility is discussed.

Keywords: iterated belief revision, information, qualitative Bayes rule, plausibility ordering

1. Introduction

Belief revision is a central topic in several fields. In game theory, belief revision is the main building block of two widely used solution concepts for dynamic (or extensive) games, namely perfect Bayesian equilibrium (see, for example, Battigalli [2], Bonanno [5] and Fudenberg and Tirole [10]) and sequential equilibrium (Kreps and Wilson [16]). The idea behind these solution concepts is that, during the play of the game, a player should revise his beliefs by using Bayes' rule "as long as possible". Thus if an information set has been reached that had positive prior probability, then beliefs at that information set are obtained by using Bayes' rule (with the information being represented by the set of nodes in the information set under consideration). If an information set is reached that had zero prior probability, then new beliefs are formed more or less arbitrarily, but from that point onwards these new beliefs must be used in conjunction with Bayes' rule, unless further information is received that is inconsistent with those revised beliefs. In computer science the theory of belief revision pioneered by Alchourrón *et al* [1] (known as the AGM theory) has been studied extensively (for an overview see Gärdenfors [11]). While in game theory beliefs are typically represented by a probability distribution over a set of states and belief revision is modeled using Bayes' rule, in the AGM theory beliefs are modeled syntactically as sets of formulas, called belief sets, in a given

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language. Information is thought of as a formula in this language and belief revision is modeled as an operation that transforms a belief set into a new belief set that incorporates the information. Within the AGM tradition the issue of iterated belief revision has recently received considerable attention (see, for example, Nayak *et al* [19]).

In this paper we propose a simple modal logic for *iterated* belief revision, extending the two-period framework of Bonanno [6]. For every date $t \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) we postulate a belief operator B_t and an information operator I_t. The interaction of information and belief over time is captured by several axioms. We start with three simple axioms and show that the corresponding logic is sound and complete with respect to the class of Kripke structures that satisfy the iterative version of the qualitative content of Bayes' rule. We then consider several strengthenings of this logic and study their properties. For the strongest of these logics we show that every belief revision history can be rationalized in terms of a plausibility ordering of the set of states.

In the next section we begin with an example that illustrates the structures analyzed in the remainder of the paper.

2. A motivating example

A doctor examines a patient who reported an outbreak of skin rashes. The patient claims not to have made any changes in his diet and gives the doctor a list of medications that he has been taking for several years. Based on her experience, the doctor narrows down the possible causes to four: bacterial infection (B), viral infection (V), allergic reaction to food (F) and allergic reaction to medication (M). An initial assessment of the case (**Time 0**) leads the doctor to believe that it is an infection. Since she knows of no treatment for a viral infection, she prescribes antibiotics which would be effective against a bacterial infection. A few days later (**Time 1**) the patient reports that there has been no change in his symptoms. The doctor treats this report as information that it is not a case of bacterial infection and becomes convinced that it is a viral infection. She informs the patient that, unfortunately, there are no drugs that would be effective against a viral infection. The patient requests a blood test. A positive result would confirm the presence of an infection, while a negative result would rule out an infection. The doctor yields to the patient's request and a few days later (**Time 2**) the lab reports a negative result to the blood test. Based on this information, the doctor reaches the conclusion that the patient must have developed a sensitivity to one of the drugs and advises the patient to stop taking all his medications. A few days later (**Time 3**) the patient reports, once again, that there has been no change in his symptoms. The doctor then concludes that it must be an allergic reaction to food and instructs the patient to keep a detailed food diary.

In this example the doctor's beliefs evolve over time in response to new information. The history of the doctor's beliefs can be represented using a sequence of Kripke structures (Kripke [17]): the set of states, or possible worlds, is $\Omega = \{\mathrm{B,V,M,F}\}$ (where B denotes bacterial infection, V viral infection, M medication allergy and F food allergy) and at every date t the doctor's beliefs can be represented by a binary relation $\mathcal{B}_t$ on Ω. For every state $\omega \in \Omega$ let $\mathcal{B}_t(\omega) = \{\omega' \in \Omega : \omega\mathcal{B}_t\omega'\}$, that is, $\mathcal{B}_t(\omega)$ is the set of states that–at time t–the doctor considers possible when the true state is ω. In the above example, taking the true state to be F, the actual evolution of the doctor's belief is as follows: $\mathcal{B}_0(\mathrm{F}) = \{\mathrm{B,V}\}$, $\mathcal{B}_1(\mathrm{F}) = \{\mathrm{V}\}$, $\mathcal{B}_2(\mathrm{F}) = \{\mathrm{M}\}$ and $\mathcal{B}_3(\mathrm{F}) = \{\mathrm{F}\}$. In general, the binary relations $\{\mathcal{B}_t\}_{t\in\mathbb{N}}$ describe the possible evolutions of the doctor's beliefs in every possible case, that is, whatever the true state.

Syntactically, let B_t be the belief operator at time t, so that the interpretation of $B_t\phi$ is "at time t the individual believes that ϕ". If ω is a state and ϕ a formula, we denote by $\omega \vDash \phi$ the fact that ϕ is true at state ω. The truth of the formula $B_t\phi$ at state ω is determined as usual: $\omega \vDash B_t\phi$ if and only, for every ω' such that $\omega\mathcal{B}_t\omega'$, $\omega' \vDash \phi$, that is, if ϕ is true at every state that, at date t, the individual considers possible at state ω (if we denote the truth set of ϕ by $||\phi||$, the truth condition for $B_t\phi$ can be also be written as follows: $\omega \vDash B_t\phi$ if and only if $\mathcal{B}_t(\omega) \subseteq ||\phi||$). For example, if ϕ is the proposition "the patient has an infection" then, in our example, it is true at – and only at – states B and V. If the true state is F then at that state it is false that the patient has an infection and yet it is true that at date 0 the doctor believes that the patient has an infection (F$\vDash \neg\phi$ and F$\vDash B_0\phi$).

Changes in the doctor's beliefs are brought about by the receipt of new information. It seems natural to represent the flow of information over time in the same way in which we represent beliefs, namely by means of a sequence of binary relations $\mathcal{I}_t$, for every date t. We propose to model information in a way which is reminiscent of the notion of "only knowing" (Levesque [18]). Intuitively, we interpret "I am informed that ϕ" to mean "all I am told is ϕ". We capture this interpretation of information by means of the following validation rule. Let I_t be the time t information operator, so that the interpretation of $I_t\phi$ is "at time t the individual is informed that ϕ". Then we set $\omega \vDash I_t\phi$ if and only if two conditions hold: (1) for every ω' such that $\omega\mathcal{I}_t\omega'$, $\omega' \vDash \phi$, and (2) for every $\omega' \in \Omega$, if $\omega' \vDash \phi$ then $\omega\mathcal{I}_t\omega'$. That is, $I_t\phi$ is true at state ω if the set of states reachable from ω by means of the relation $\mathcal{I}_t$ **coincides** with the truth set of ϕ: $\mathcal{I}_t(\omega) = ||\phi||$.[1]

The doctor's beliefs at time $t+1$ are the result of the interaction between her beliefs at time t and the information received at time $t+1$. In our example, at time 0 the doctor believes that the patient has a (bacterial or viral) infection and, at time 1, she is informed that the patient does not have a bacterial infection. Combining the two, the doctor modifies her beliefs and comes to the new belief that the patient has a viral infection. In a probabilistic setting, this new belief is what would be required by Bayes' rule. The interaction of old beliefs and new information in our example is shown in Figure 1. In all the figures we represent a binary relation $R \subseteq \Omega\times\Omega$ as follows: (1) if a rounded rectangle encloses a set of states then, for any two states ω and ω' in that rectangle, $\omega R\omega'$ and (2) if there is an arrow from a state ω to a rounded rectangle, then for any state ω' in that rectangle, $\omega R\omega'$. For example, in Figure 1 we have that

$$\mathcal{B}_0 = \{(\mathrm{B,B}),(\mathrm{B,V}),(\mathrm{V,B}),(\mathrm{V,V}),(\mathrm{F,B}),(\mathrm{F,V}),(\mathrm{M,B}),(\mathrm{M,V})\},$$

$$\mathcal{I}_1 = \{(\mathrm{B,B}),(\mathrm{V,V}),(\mathrm{V,F}),(\mathrm{V,M}),(\mathrm{F,V}),(\mathrm{F,F}),(\mathrm{F,M}),(\mathrm{M,V}),(\mathrm{M,F}),(\mathrm{M,M})\},$$

$$\mathcal{B}_1 = \{(\mathrm{B,B}),(\mathrm{V,V}),(\mathrm{F,V}),(\mathrm{M,V})\}, \text{ etc.}$$

In this example we have represented information at every date by means of an equivalence relation, which implies that information is truthful or correct. *Our results, however, do not require such an assumption:* in general it is possible for "information" to be incorrect (hence misleading, if believed). For example, the lack of a reaction to antibiotics can be interpreted by the doctor as information that the patient does not have a bacterial infection, while–in reality–the patient may have been infected by a bacterium that has developed resistance to antibiotics.

It is worth noting that, while the first piece of information received by the doctor ("it is not a bacterial infection") does not cause surprise–since it is compatible with the doctor's initial belief (it could be a viral infection)–later pieces of information do cause surprise, since they contradict the doctor's beliefs (when the true state is F).

In Figure 1, information at time 1 represents the outcome of the antibiotic treatment, so that the information is that the true state is either V or F or M–if the treatment was not effective–while it would have revealed the true

[1] Thus our information operator plays the role of the conjunction of the two operators K and O proposed by Levesque [18] who interprets $K\phi$ as "the individual knows that ϕ" and $O\phi$ as "the individual *only* knows that ϕ".

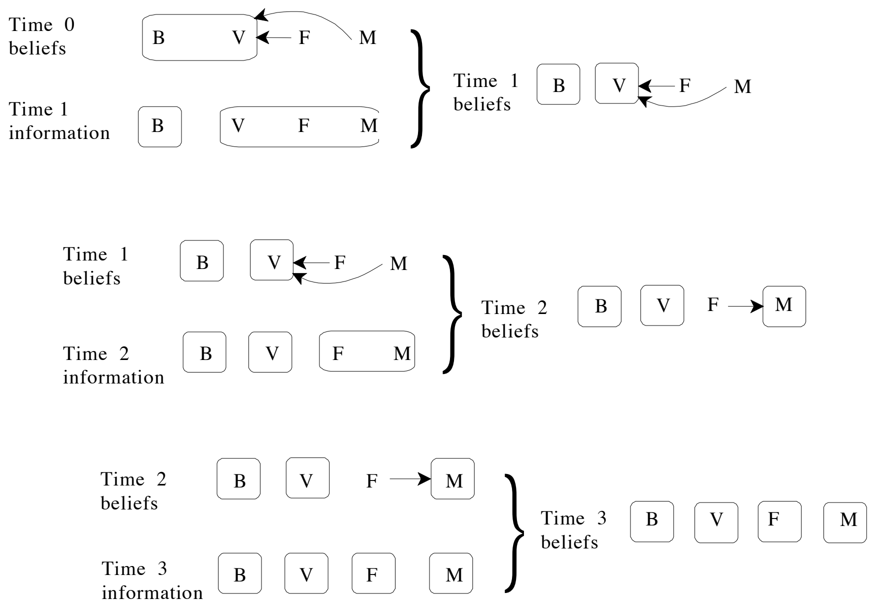

Figure 1. The interaction of the doctor's beliefs and information over time

state to be B if it had been effective. Information at time 2 represents the outcome of a blood test, revealing that the true state is either F or M, if negative, or the presence of an infection if positive. In the latter case we have represented the *cumulative* information given by the outcome of the antibiotic treatment together with the information given by the outcome of the blood test, thus enabling the doctor to distinguish between states B and V. Alternatively, we could have represented at date 2 the *new* information conveyed by the outcome of the blood test, which–if positive–would merely indicate that there is an infection, that is, that the state is either B or V. A similar observation can be made concerning information at time 3. Thus an alternative representation of the example discussed above could be as shown in Figure 2. Note, however, that when belief revision obeys the axioms introduced in the next section, the evolution of beliefs would be the same in the two representations of information.

In the next section we propose a logic that is sound and complete with respect to the class of Kripke structures that satisfy the qualitative version of Bayes' rule (the structures illustrated in Figures 1 and 2 belong to this class). We then consider several strengthenings of that logic and discuss their relationship to the AGM belief revision theory (see Alchourrón *et al* [1]).

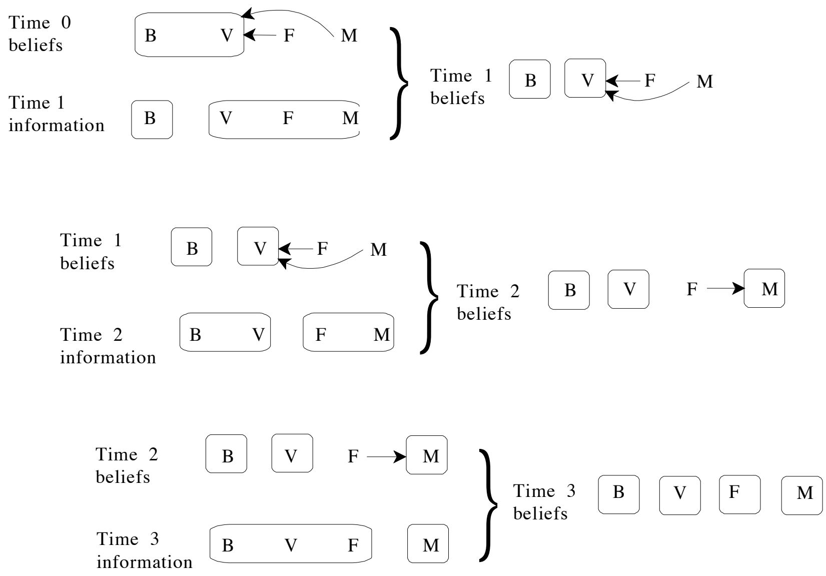

Figure 2. The evolution of the doctor's beliefs when information is not cumulative

Finally, we show that when information becomes more refined over time (for example, this is the case in the structure of Figure 1) the individual's belief revision can be rationalized in terms of a plausibility ordering over the set of states: at every date the individual considers possible all and only those states that are most plausible among the ones that are compatible with the new information.

3. A logic for iterated belief revision

Let $\mathbb{N}$ be the set of non-negative integers. We consider a modal propositional logic based on the following operators: a belief operator B_t and an information operator I_t, for every date $t \in \mathbb{N}$, and a "global" operator A.[2] The intended interpretation is as follows:

$I_t\phi$	at time t all the individual is informed of is that ϕ
$B_t\phi$	at time t (after revising his earlier beliefs in light of the information just received) the individual believes that ϕ
$A\phi$	it is true at every state that ϕ.

[2]For a discussion of the global (or universal) modality see Goranko and Passy [12].

The formal language is built in the usual way from a countable set S of atomic propositions, the connectives $\neg$ (for "not") and $\vee$ (for "or") and the modal operators.[3] Thus the set Φ of formulas is defined inductively as follows: $q \in \Phi$ for every atomic proposition $q \in S$, and if $\phi, \psi \in \Phi$ then all of the following belong to Φ: $\neg\phi$, $\phi \vee \psi$, $A\phi$ and, for every $t \in \mathbb{N}$, $B_t\phi$ and $I_t\phi$.

We denote by $\mathbb{L}$ the logic determined by the following axioms and rules of inference.

AXIOMS:

1. All propositional tautologies.
2. Axiom K for A and B_t (for every $t \in \mathbb{N}$):

$$\begin{array}{ll} \mathrm{K}_A: & A\phi \wedge A(\phi \to \psi) \to A\psi \\ \mathrm{K}_B: & B_t\phi \wedge B_t(\phi \to \psi) \to B_t\psi. \end{array}$$

3. S5 axioms for A:

$$\begin{array}{ll} \mathrm{T}_A: & A\phi \to \phi \\ 5_A: & \neg A\phi \to A\neg A\phi. \end{array}$$

4. For every $t \in \mathbb{N}$, inclusion axiom for B_t (note the absence of an analogous axiom for I_t):

$$\text{Incl:} \quad A\phi \to B_t\phi.$$

5. Axioms to capture the non-standard semantics for I_t:

$$\begin{array}{ll} \mathrm{I}_1: & (I_t\phi \wedge I_t\psi) \to A(\phi \leftrightarrow \psi) \\ \mathrm{I}_2: & A(\phi \leftrightarrow \psi) \to (I_t\phi \leftrightarrow I_t\psi). \end{array}$$

RULES OF INFERENCE:

1. Modus Ponens (MP): $\frac{\phi, \quad \phi \to \psi}{\psi}$
2. Necessitation for A (Nec_A): $\frac{\phi}{A\phi}$.

[3] See, for example, Blackburn *et al* [3]. The connectives $\wedge$ (for "and"), $\to$ (for "if ... then ...") and $\leftrightarrow$ (for "if and only if") are defined as usual: $\phi \wedge \psi = \neg(\neg\phi \vee \neg\psi)$, $\phi \to \psi = \neg\phi \vee \psi$ and $\phi \leftrightarrow \psi = (\phi \to \psi) \wedge (\psi \to \phi)$.

Remark 1. *We have allowed $I_t\phi$ to be a well-formed formula for every formula ϕ. As pointed out by Friedman and Halpern [9], this may be problematic. For example, it is not clear how one could be informed of a contradiction. Furthermore, one might want to restrict information to facts by not allowing $I_t\phi$ to be a well-formed formula if ϕ contains any of the modal operators B_t and I_t. Without that restriction, in principle we admit situations like the following: at time t the individual believes that ϕ and is later informed that he did not believe that ϕ: $B_t\phi \wedge I_{t+1}\neg B_t\phi$. It is not clear how such a situation could arise. However, since our results remain true–whether or not we impose the restriction- -we have chosen to follow the more general approach. The undesirable situations can then be eliminated by imposing suitable axioms.*[4]

On the semantic side, a *frame* is a collection $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}\rangle$ where Ω is a set of *states* and, for every $t \in \mathbb{N}$, $\mathcal{B}_t$ and $\mathcal{I}_t$ are binary relations on Ω, whose interpretation is as follows:

$\alpha\mathcal{B}_t\beta$	at time t and state α the individual considers state β possible
$\alpha\mathcal{I}_t\beta$	at state α, state β is compatible with the information received at time t.

Let $\mathcal{B}_t(\omega) = \{\omega' \in \Omega : \omega\mathcal{B}_t\omega'\}$ denote the set of states that, at date t, the individual considers possible at state ω. Similarly, $\mathcal{I}_t(\omega) = \{\omega' \in \Omega : \omega\mathcal{I}_t\omega'\}$.

As usual the connection between syntax and semantics is given by the notion of model. Given a frame $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}\rangle$, a *model* is obtained by adding a *valuation* $V : S \to 2^\Omega$ (where 2^Ω denotes the set of subsets of Ω, usually called *events*) which associates with every atomic proposition $p \in S$ the set of states at which p is true.[5] The truth of an arbitrary formula at a state is then defined inductively as follows ($\omega \vDash \phi$ denotes that formula ϕ is

[4]For example, contradictory information is ruled out by the axiom $I_t\phi \to \neg A\neg\phi$. The axiom $B_t\phi \to \neg I_{t+1}\neg B_t\phi$ rules out being informed that earlier one did not believe ϕ when, in fact, one did. Another example of a problematic situation is represented by the formula $I_t(\phi \wedge \neg B_t\phi)$ (the individual is informed that ϕ and that he will not believe ϕ). Such a situation cannot arise under standard assumptions about beliefs. For example, if one imposes the Acceptance axiom ($I_t\phi \to B_t\phi$: see below), consistency of beliefs ($B_t\psi \to \neg B_t\neg\psi$) and positive introspection of beliefs ($B_t\psi \to B_tB_t\psi$) then it can be shown that, in every model, for every state ω, date t and formula ϕ, $\omega \vDash \neg I_t(\phi \wedge \neg B_t\phi)$.

[5]Note that by making the truth of atomic propositions depend on the state only, rather than on the state and time, we restrict ourselves to situations of *belief revision*, where the objective description of the world does not change over time: only the epistemic state of the individual changes. The alternative case, where the truth of the atomic propositions is allowed to change over time, is known in the computer science literature as *belief update* (see Katsuno and Mendelzon [15]).

true at state ω; $\|\phi\|$ is the truth set of ϕ, that is, $\|\phi\| = \{\omega \in \Omega : \omega \vDash \phi\}$):

if q is an atomic proposition, $\omega \vDash q$ if and only if $\omega \in V(q)$,
$\omega \vDash \neg\phi$ if and only if $\omega \nvDash \phi$,
$\omega \vDash \phi \vee \psi$ if and only if either $\omega \vDash \phi$ or $\omega \vDash \psi$ (or both),
$\omega \vDash B_t\phi$ if and only if $\mathcal{B}_t(\omega) \subseteq \|\phi\|$,
$\omega \vDash I_t\phi$ if and only if $\mathcal{I}_t(\omega) = \|\phi\|$,
$\omega \vDash A\phi$ if and only if $\|\phi\| = \Omega$.

REMARK 2. *Note that, while the truth condition for $B_t\phi$ is the standard one, the truth condition of $I_t\phi$ is unusual in that the requirement is $\mathcal{I}_t(\omega) = \|\phi\|$ rather than merely $\mathcal{I}_t(\omega) \subseteq \|\phi\|$. This is what establishes the similarity between our information operator and the "only knowing" operator discussed in the literature (see Levesque [18]).*

We say that a formula ϕ is *valid in a model* if $\omega \vDash \phi$ for all $\omega \in \Omega$, that is, if ϕ is true at every state. A formula ϕ is *valid in a frame* if it is valid in every model based on that frame. A logic is *sound* with respect to a class of frames if every theorem of the logic is valid in every frame in that class; it is *complete* with respect to a class of frames if every formula which is valid in every frame in that class is provable in the logic (that is, it is a theorem).

REMARK 3. *Note that from (Nec_A) and (Incl) one obtains necessitation for B_t as a derived rule of inference: $\frac{\phi}{B_t\phi}$. Furthermore, from necessitation and axiom K one obtains the following derived rule of inference for both A and B_t (usually referred to as rule RK): $\frac{\phi\to\psi}{A\phi\to A\psi}$ and $\frac{\phi\to\psi}{B_t\phi\to B_t\psi}$. On the other hand, the necessitation rule for I_t is **not** a rule of inference of logic $\mathbb{L}$. Indeed necessitation for I_t is not validity preserving[6]; neither is rule RK for I_t.[7]*

Note that, despite the non-standard validation rule, axiom K for I_t, namely $I_t\phi \wedge I_t(\phi \to \psi) \to I_t\psi$, is trivially valid in every frame.[8] It follows from the completeness part of Proposition 4 below that axiom K for I_t is a theorem of logic $\mathbb{L}$ (a syntactic proof is also easily obtained).

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[6]If ϕ is a valid formula, then $\|\phi\| = \Omega$. Let $\omega \in \Omega$ be a state where $\mathcal{I}_t(\omega) \neq \Omega$. Then $\omega \nvDash I_t\phi$ and therefore $I_t\phi$ is not valid.

[7]That is, from the validity of $\phi \to \psi$ one cannot infer the validity of $I_t\phi \to I_t\psi$. To see this, consider the following model: $\Omega = \{\alpha, \beta\}$, $\mathcal{I}_t(\alpha) = \{\alpha\}$, $\mathcal{I}_t(\beta) = \{\beta\}$, $\|p\| = \{\alpha\}$ and $\|q\| = \Omega$. Then $\|p \to q\| = \Omega$, $\|I_tp\| = \{\alpha\}$, $\|I_tq\| = \varnothing$ and thus $\|I_tp \to I_tq\| = \{\beta\} \neq \Omega$.

[8]PROOF. Fix a frame, an arbitrary model and a state ω. For it to be the case that $\omega \vDash I_t\phi \wedge I_t(\phi \to \psi)$ we need $\mathcal{I}_t(\omega) = \|\phi\|$ and $\mathcal{I}_t(\omega) = \|\phi \to \psi\|$. Now, $\|\phi \to \psi\| = \|\neg\phi \vee \psi\| = \|\neg\phi\| \cup \|\psi\|$ and therefore we need the equality $\|\phi\| = \|\neg\phi\| \cup \|\psi\|$ to be satisfied. This requires $\|\phi\| = \|\psi\| = \Omega$. Thus if $\mathcal{I}_t(\omega) = \|\phi\| = \|\psi\| = \Omega$, then $\omega \vDash I_t(\phi \to \psi) \wedge I_t\phi \wedge I_t\psi$. In every other case, $\omega \nvDash I_t\phi \wedge I_t(\phi \to \psi)$ and therefore the formula $I_t\phi \wedge I_t(\phi \to \psi) \to I_t\psi$ is trivially true at ω.

The following proposition is an extension of the analogous result in the two-period framework of Bonanno [6]. An outline of the proof is given in the Appendix.

Proposition 4. *Logic* $\mathbb{L}$ *is sound and complete with respect to the class of all frames* $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}\rangle$.

We now consider extensions of logic $\mathbb{L}$. The first extension, denoted by $\mathbb{L}_{QBR}$, is obtained by adding to $\mathbb{L}$ the following axioms:

Qualified Acceptance (QA):	$(\neg B_t\neg\phi \wedge I_{t+1}\phi) \to B_{t+1}\phi$
Persistence (P) (or No Drop):	$(\neg B_t\neg\phi \wedge I_{t+1}\phi) \to (B_t\psi \to B_{t+1}\psi)$
Minimality (M) (or No Add):	$(I_{t+1}\phi \wedge B_{t+1}\psi) \to B_t(\phi \to \psi)$.

One of the axioms of the AGM theory of belief revision is the so-called Success or Acceptance axiom, which requires that information be believed, that is, that it be incorporated in the revised beliefs. Our Qualified Acceptance axiom is a weakening of this, in that it requires the individual who has been informed that ϕ to believe ϕ only if, before the receipt of information, he considered ϕ possible (that is, he did not believe $\neg\phi$). The Persistence axiom says that if the individual is informed of something that he previously considered possible, then he continues to believe everything that he believed before, that is, he cannot drop any beliefs that he had then. Finally, the Minimality axiom states that beliefs should be revised in a minimal way, in the sense that no new beliefs should be added unless they are implied by the old beliefs and the information received.[9]

Definition 5. *A* QBR frame *is a frame* $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}\rangle$ *that satisfies the following property, which we call the* Qualitative Bayes Rule*:* $\forall\omega \in \Omega$, $\forall t \in \mathbb{N}$,

$$\text{if } \mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega) \neq \varnothing \text{ then } \mathcal{B}_{t+1}(\omega) = \mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega). \qquad (QBR)$$

Property QBR says that if at a state the information received at time $t+1$ is compatible with the beliefs the individual had at time t, in the sense that there are states that he considered possible at date t and are consistent with the information received at date $t+1$, then the states that he considers

[9]For a more in-depth discussion of these three axioms (for example concerning the seemingly problematic derivability of $(I_{t+1}\phi \wedge B_{t+1}\phi) \to B_t(\phi \to B_{t+1}\phi)$ from Minimality and positive introspection of beliefs) see [6] (pp. 201-202).

As noted there, the Minimality axiom restricts the new beliefs only when the information received is not surprising, that is, only if $\omega \vDash \neg B_t\neg\phi$.

possible according to the revised beliefs at date $t+1$ are precisely those states. For example, the frames illustrated in Figures 1 and 2 of Section 2 satisfy QBR (vacuously after date 1).

In a probabilistic setting, let $P_{t,\omega}$ be the probability measure over Ω representing the individual's beliefs at date t and state ω, let $F \subseteq \Omega$ be an event representing the information received by the individual at date $t+1$ and let $P_{t+1,\omega}$ be the posterior probability measure representing the revised beliefs at date $t+1$ and state ω. Bayes' rule requires that, if $P_{t,\omega}(F) > 0$, then, for every event $E \subseteq \Omega$, $P_{t+1,\omega}(E) = \frac{P_{t,\omega}(E \cap F)}{P_{t,\omega}(F)}$. Bayes' rule thus implies the following (where $supp(P)$ denotes the support of the probability measure P):

$$\text{if } \; supp(P_{t,\omega}) \cap F \neq \varnothing, \text{ then } \; supp(P_{t+1,\omega}) = supp(P_{t,\omega}) \cap F.$$

If we set $\mathcal{B}_t(\omega) = supp(P_{t,\omega})$, $F = \mathcal{I}_{t+1}(\omega)$ and $\mathcal{B}_{t+1}(\omega) = supp(P_{t+1,\omega})$ then we get the Qualitative Bayes Rule as stated above. Thus in a probabilistic setting the sentence "the individual believes ϕ" would be interpreted as "the individual assigns probability 1 to the event $||\phi||$".

The following proposition, proved in the Appendix, is an extension of the two-period framework of Bonanno [6].

Proposition 6. *Logic $\mathbb{L}_{QBR}$ is sound and complete with respect to the class of frames $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}} \rangle$ that satisfy the Qualitative Bayes Rule.*

We now consider stronger logics than $\mathbb{L}_{QBR}$. Let $\mathbb{L}_{WAGM}$ ($WAGM$ stands for "Weak AGM") be the logic obtained by adding to $\mathbb{L}$ the following axioms:

Consistent Acceptance (CA):	$(I_t\phi \wedge \neg A\neg\phi) \rightarrow B_t\phi$
Persistence (P):	$(\neg B_t\neg\phi \wedge I_{t+1}\phi) \rightarrow (B_t\psi \rightarrow B_{t+1}\psi)$
Minimality (M):	$(I_{t+1}\phi \wedge B_{t+1}\psi) \rightarrow B_t(\phi \rightarrow \psi)$
Weak Consistency of beliefs (WC)	$(I_t\phi \wedge \neg A\neg\phi) \rightarrow (B_t\psi \rightarrow \neg B_t\neg\psi)$

The Consistent Acceptance axiom says that if the agent is informed of ϕ and ϕ is a consistent formula ($\neg A\neg\phi$) then he believes ϕ (even if he previously believed $\neg\phi$). By axiom *Incl*, Qualified Acceptance can be derived from Consistent Acceptance[10] and therefore $\mathbb{L}_{WAGM}$ is an extension

[10] Proof.

1.	$A\neg\phi \rightarrow \mathcal{B}_t\neg\phi$	Axiom *Incl*
2.	$\neg\mathcal{B}_t\neg\phi \rightarrow \neg A\neg\phi$	1, PL
3.	$(\neg\mathcal{B}_t\neg\phi \wedge I_{t+1}\phi) \rightarrow (\neg A\neg\phi \wedge I_{t+1}\phi)$	2, PL
4.	$(\neg A\neg\phi \wedge I_{t+1}\phi) \rightarrow B_{t+1}\phi$	Axiom CA
5.	$(\neg\mathcal{B}_t\neg\phi \wedge I_{t+1}\phi) \rightarrow B_{t+1}\phi$	3, 4, PL. ∎

of $\mathbb{L}_{QBR}$. Axiom WC (which stands for "Weak Consistency of beliefs") says that if the agent is presented with consistent information then his beliefs are consistent, in the sense that he cannot simultaneously believe something and its negation.

Definition 7. *A WAGM frame is a frame* $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t \in \mathbb{N}} \rangle$ *that satisfies the following properties:* $\forall t \in \mathbb{N}$, $\forall \omega \in \Omega$,

(1) if $\mathcal{I}_t(\omega) \neq \varnothing$ *then* $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$,
(2) if $\mathcal{I}_t(\omega) \neq \varnothing$ *then* $\mathcal{B}_t(\omega) \neq \varnothing$,
(3) the Qualitative Bayes Rule (if $B_t(\omega) \cap I_{t+1}(\omega) \neq \varnothing$
then $B_{t+1}(\omega) = B_t(\omega) \cap I_{t+1}(\omega)$*).*

For example, the frames illustrated in Figures 1 and 2 of Section 2 are WAGM frames. Clearly every WAGM frame is a QBR frame, but not *vice versa.*

The following proposition is proved in the Appendix.

Proposition 8. *Logic* $\mathbb{L}_{WAGM}$ *is sound and complete with respect to the class of WAGM frames.*

Our next result shows that logic $\mathbb{L}_{WAGM}$ captures the spirit of the AGM theory in the sense that it satisfies the basic set of AGM postulates whenever information is not contradictory. Indeed, as pointed out by Friedman and Halpern [9], it is not clear how information could consist of a contradiction.

In order to establish the relationship between logic $\mathbb{L}_{WAGM}$ and the AGM theory of belief revision we first need to recall the AGM postulates. Let Γ be the set of formulas in a propositional language. Given a subset $\Sigma \subseteq \Gamma$, its PL-deductive closure $[\Sigma]^{PL}$ (where 'PL' stands for 'Propositional Logic') is defined as follows: $\psi \in [\Sigma]^{PL}$ if and only if there exist $\phi_1, ..., \phi_n \in \Sigma$ such that $(\phi_1 \wedge ... \wedge \phi_n) \to \psi$ is a truth-functional tautology (that is, a theorem of Propositional Logic). A *belief set* is a set $K \subseteq \Gamma$ such that $K = [K]^{PL}$. A belief set K is consistent if $K \neq \Gamma$ (equivalently, if there is no formula ϕ such that both ϕ and $\neg\phi$ belong to K). Given a belief set K (thought of as the initial beliefs of the individual) and a formula ϕ (thought of as a new piece of information), the *revision of* K *by* ϕ, denoted by K^*_ϕ, is a subset of Γ that satisfies the following conditions, known as the AGM postulates:

(K*1) K^*_ϕ is a belief set
(K*2) $\phi \in K^*_\phi$
(K*3) $K^*_\phi \subseteq [K \cup \{\phi\}]^{PL}$
(K*4) if $\neg\phi \notin K$, then $[K \cup \{\phi\}]^{PL} \subseteq K^*_\phi$
(K*5) $K^*_\phi = \Gamma$ if and only if ϕ is a contradiction

(K*6) if $\phi \leftrightarrow \psi$ is a tautology then $K^*_\phi = K^*_\psi$

(K*7) $K^*_{\phi\wedge\psi} \subseteq \left[K^*_\phi \cup \{\psi\}\right]^{PL}$

(K*8) if $\neg\psi \notin K^*_\phi$, then $\left[K^*_\phi \cup \{\psi\}\right]^{PL} \subseteq K^*_{\phi\wedge\psi}$

The set of postulates (K*1) through (K*6) is called the *basic set* of postulates for belief revision (Gärdenfors, [11] p. 55). The next proposition, proved in the Appendix, shows that every model of logic $\mathbb{L}_{WAGM}$ satisfies the basic set of AGM postulates.

PROPOSITION 9. *Fix an arbitrary model based on a WAGM frame (see Definition 7). Fix arbitrary state ω and date t and let $K = \{\psi : \omega \vDash B_t\psi\}$. Suppose that there is a formula ϕ such that $\omega \vDash I_{t+1}\phi$ and $\omega' \vDash \phi$ for some $\omega' \in \Omega$ (so that ϕ is a consistent formula). Define $K^*_\phi = \{\psi : \omega \vDash B_{t+1}\psi\}$. Then K is a belief set and K^*_ϕ satisfies the basic AGM postulates (K*1) to (K*6).*

AGM postulates (K*7) and (K*8), require that the revision of K that includes both information ϕ and information ψ (that is, $K^*_{\phi\wedge\psi}$) ought to be the same as the expansion of K^*_ϕ by ψ, so long as ψ does not contradict the beliefs in K^*_ϕ. In our framework we are able to model, at every date and state, only the information that is actually received by the individual and cannot capture the counterfactual of how the individual would have modified his beliefs if he had received a different piece of information. Thus we cannot compare the revised beliefs that the individual holds after first receiving information ϕ and subsequently information ψ with the beliefs he would have had if he had been simultaneously informed of both ϕ and ψ. One way to capture the entire set of AGM postulates is to consider a branching-time framework which allows the comparison of different belief revisions along different branches. This is done in Bonanno [7]. Here we follow an indirect route through the notion of plausibility. In the literature it has been shown that there is an equivalence between the full set of AGM postulates and the notion of a plausibility ordering of the set of states (see Grove [13] and Board [4]).

A *plausibility ordering of* Ω is a binary relation $\precsim$ on Ω that is complete ($\forall \omega, \omega' \in \Omega$, either $\omega \precsim \omega'$ or $\omega' \precsim \omega$) and transitive (if $\omega \precsim \omega'$ and $\omega' \precsim \omega''$ then $\omega \precsim \omega''$). Given a plausibility ordering $\precsim$ of Ω and a subset $X \subseteq \Omega$, we denote by $\min_{\precsim} X$ the set $\{\omega \in X : \omega \precsim \omega' \text{ for all } \omega' \in X\}$.

DEFINITION 10. *Given a frame $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}\rangle$ and a state $\omega \in \Omega$, we call the sequence $\{\mathcal{B}_t(\omega), \mathcal{I}_t(\omega)\}_{t\in\mathbb{N}}$ a* belief revision history. *A plausibility relation*

$\precsim$ *on* Ω rationalizes *the belief revision history* $\{\mathcal{B}_t(\omega), \mathcal{I}_t(\omega)\}_{t\in\mathbb{N}}$ *if, for every* $t \in \mathbb{N}$, $\mathcal{B}_t(\omega) = \min_{\precsim} \mathcal{I}_t(\omega)$.

Thus a belief revision history is rationalized by a plausibility ordering if, at every date, the set of states that the individual considers possible is the set of most plausible states among the ones that are compatible with the information received.

Definition 11. *A belief revision history* $\{\mathcal{B}_t(\omega), \mathcal{I}_t(\omega)\}_{t\in\mathbb{N}}$ *is* consistent, successful and information-refined *if it satisfies the following properties:* $\forall t \in \mathbb{N}$,

(1) $\mathcal{B}_t(\omega) \neq \varnothing$,
(2) $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$,
(3) $\mathcal{I}_{t+1}(\omega) \subseteq \mathcal{I}_t(\omega)$.

For example, in the frame illustrated in Figure 1 of Section 2, for every state the corresponding belief revision history is consistent, successful and information-refined.

The following proposition is proved in the Appendix.

Proposition 12. *Given a belief revision history* $\mathcal{H} = \{\mathcal{B}_t(\omega), \mathcal{I}_t(\omega)\}_{t\in\mathbb{N}}$ *which is consistent, successful and information-refined there exists a plausibility relation* $\precsim$ *that rationalizes* $\mathcal{H}$ *if and only if* $\mathcal{H}$ *is Qualitatively Bayesian, that is, if and only if,* $\forall t \in \mathbb{N}$, *if* $\mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega) \neq \varnothing$ *then* $\mathcal{B}_{t+1}(\omega) = \mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega)$.

With the help of the above proposition we can define a class of frames with the property that every belief revision history can be rationalized in terms of a plausibility relation.

Definition 13. *A SAGM frame (SAGM stands for "Strong AGM") is a frame* $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}\rangle$ *that satisfies the following properties:* $\forall t \in \mathbb{N}$, $\forall \omega \in \Omega$,

(1) $\mathcal{B}_t(\omega) \neq \varnothing$,
(2) $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$,
(3) $\mathcal{I}_{t+1}(\omega) \subseteq \mathcal{I}_t(\omega)$,
(4) the Qualitative Bayes Rule (if $B_t(\omega) \cap I_{t+1}(\omega) \neq \varnothing$
then $B_{t+1}(\omega) = B_t(\omega) \cap I_{t+1}(\omega)$*).*

Clearly, every $SAGM$ frame is a $WAGM$ frame but not *vice versa.*

The following result is a Corollary of Proposition 12.

COROLLARY 14. *Let $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}\rangle$ be a SAGM frame. Then, for every $\omega \in \Omega$, there exists a plausibility relation $\precsim$ on Ω that rationalizes the belief history $\{\mathcal{B}_t(\omega), \mathcal{I}_t(\omega)\}_{t\in\mathbb{N}}$.*

For example, the frame illustrated in Figure 1 of Section 2 is a *SAGM* frame and, indeed, every belief revision history $\{\mathcal{B}_t(\omega), \mathcal{I}_t(\omega)\}_{t\in\{0,1,2,3\}}$ with $\omega \in \{\mathrm{B,V,M,F}\}$ is rationalized by the following plausibility ordering (which, in this case, is independent of the state):

$\precsim$ =
$\{(\mathrm{B,B}),(\mathrm{B,V}),(\mathrm{B,F}),(\mathrm{B,M}),(\mathrm{V,B}),(\mathrm{V,V}),(\mathrm{V,F}),(\mathrm{V,M}),(\mathrm{M,F}),(\mathrm{M,M}),(\mathrm{F,F})\}$.

The class of *SAGM* frames corresponds (in the sense of frame correspondence: see Blackburn *et al* [3]) to the following logic, which we call $\mathbb{L}_{SAGM}$: the basic logic $\mathbb{L}$ with the addition of the following axioms:

Consistency of beliefs (D_B):	$B_t\phi \to \neg B_t\neg\phi$
Acceptance (A):	$I_t\phi \to B_t\phi$
Information refinement (IR):	$(I_t\phi \wedge I_{t+1}\psi) \to A(\psi \to \phi)$
Persistence (P):	$(\neg B_t\neg\phi \wedge I_{t+1}\phi) \to (B_t\psi \to B_{t+1}\psi)$
Minimality (M):	$(I_{t+1}\phi \wedge B_{t+1}\psi) \to B_t(\phi \to \psi)$

It is straightforward to show that logic $\mathbb{L}_{SAGM}$ is an extension (a strengthening) of logic $\mathbb{L}_{WAGM}$.

Note that the properties of a *SAGM* frame are sufficient but not necessary for rationalizability by means of a plausibility relation, as the example illustrated in Figure 3 shows. The frame of Figure 3 not only violates information refinement, but is not even a *QBR* frame, since it violates the Qualitative Bayes Rule: $\mathcal{B}_1(\alpha) \cap \mathcal{I}_2(\alpha) = \{\beta\} \neq \varnothing$ and yet $\mathcal{B}_2(\alpha) = \{\beta, \gamma\}$. However the belief revision history $\{\mathcal{B}_t(\alpha), \mathcal{I}_t(\alpha)\}_{t\in\{0,1,2\}}$ is rationalized by the following plausibility relation

$$\precsim \; = \{(\alpha,\alpha),(\beta,\alpha),(\beta,\beta),(\beta,\gamma),(\gamma,\alpha),(\gamma,\beta),(\gamma,\gamma)\}.$$

The frame illustrated in Figure 4, on the other hand, shows that the hypothesis of information refinement ($\mathcal{I}_{t+1} \subseteq \mathcal{I}_t$) is crucial for Corollary 14: the frame satisfies every property of Definition 13 except for information refinement (property 3) and no belief revision history can be rationalized by a plausibility relation. In fact, from the beliefs at $t = 0$ we get that β is more plausible than α and from the beliefs at $t = 2$ we get that α is more plausible than β. This example also shows that the property of information refinement cannot be weakened as follows: if $\mathcal{I}_{t+1}(\omega) \cap \mathcal{I}_t(\omega) \neq \varnothing$ then $\mathcal{I}_{t+1}(\omega) \subseteq \mathcal{I}_t(\omega)$, since the frame illustrated in Figure 4 satisfies this weaker property.

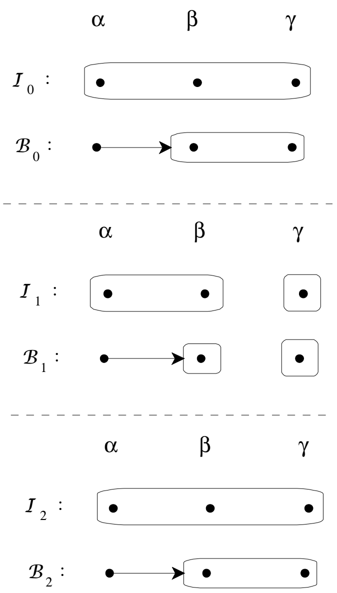

Figure 3. A non-QBR frame which is rationalizable

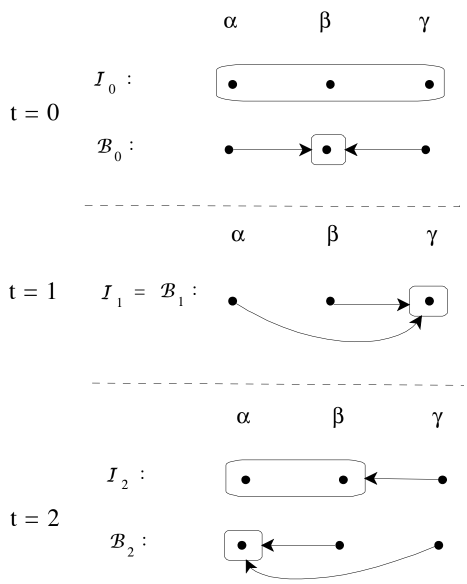

Figure 4. A QBR frame that violates information refinement

For a general characterization (which does not require information refinement) of the notion of belief revision based on plausibility more general structures are needed. Such a characterization is provided by Bonanno [7] within the framework of branching-time temporal logic.

4. Conclusion

The notions of static belief and of belief revision have been studied extensively in the literature. However, there is a surprising lack of uniformity in the two approaches. In the philosophy and logic literature, starting with Hintikka's [14] seminal contribution, the notion of *static* belief has been studied mainly within the context of modal logic. The study of *belief revision*, on the other hand, has mainly followed the AGM approach where beliefs are modeled as sets of formulas in a given syntactic language and the issue is how a belief set ought to be modified when new information, represented by a formula, becomes available. A yet different approach can be found in economics and game theory, where it is standard to represent beliefs by means of a probability measure over a set of states and belief revision is modeled using Bayes' rule. With a few exceptions, the tools of modal logic have not been explicitly employed in the analysis of the interaction of belief and information over time. In this paper we have proposed a unifying framework for static beliefs and iterated belief revision by bringing iterated belief revision under the umbrella of modal logic. For a detailed discussion of the relationship between our approach and the existing literature the reader is referred to Bonanno [6].

A. Appendix

Outline of the proof of Proposition 4. (This is an extension of the proof given for the two-date case in Bonanno [6]; thus some of the details are omitted and can be found there.) Soundness is easily proved. The completeness proof is first carried out with respect to the class of augmented frames, which are defined as follows.

Definition 15. *An* augmented frame *is a collection* $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}, \mathcal{A}\rangle$ *obtained by adding an equivalence relation* $\mathcal{A}$ *to a regular frame* $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}\rangle$ *with the additional requirement that* $\mathcal{B}_t \subseteq \mathcal{A}$ *for every* $t \in \mathbb{N}$.

Let $\mathbb{M}$ be the set of maximally consistent sets (MCS) of formulas of $\mathbb{L}$. Define the following binary relations $\mathcal{A}, \mathcal{B}_t \subseteq \mathbb{M} \times \mathbb{M}$: $\alpha\mathcal{A}\beta$ if and only if

$\{\phi : A\phi \in \alpha\} \subseteq \beta$ and $\alpha\mathcal{B}_t\beta$ if and only if $\{\phi : B_t\phi \in \alpha\} \subseteq \beta$. The relation $\mathcal{A}$ is an equivalence relation because of axioms T_A and 5_A and, for every $t \in \mathbb{N}$, $\mathcal{B}_t$ is a subrelation of $\mathcal{A}$ because of axiom $Incl$. Furthermore, the following lemma is a consequence of axioms I_1 and I_2.

LEMMA 16. *Let $\alpha, \beta \in \mathbb{M}$ be such that $\alpha\mathcal{A}\beta$ and let ϕ be a formula such that $I_t\phi \in \alpha$ and $\phi \in \beta$. Then, for every formula ψ, if $I_t\psi \in \alpha$ then $\psi \in \beta$, that is, $\{\psi : I_t\psi \in \alpha\} \subseteq \beta$.*

The definition of the relation $\mathcal{I}_t$ is more complicated, because of the non-standard validation rule for the operator I_t. Let ω_0 be an arbitrary object such that $\omega_0 \notin \mathbb{M}$, that is, ω_0 can be anything but a MCS. Define the relation $\mathcal{I}_t$ on $\mathbb{M} \cup \{\omega_0\}$ as follows: $\alpha\mathcal{I}_t\beta$ if and only if (1) $\alpha \in \mathbb{M}$ and (2) **either** $\beta \in \mathbb{M}$ and, for some formula ϕ, $I_t\phi \in \alpha$ and $\phi \in \beta$, **or** for all ϕ, $I_t\phi \notin \alpha$ and $\beta = \omega_0$.

The structure $\langle \mathbb{M} \cup \{\omega_0\}, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}, \mathcal{A}\rangle$ so defined is an augmented frame. For every $\alpha \in \mathbb{M}$, let $\mathcal{A}(\alpha) = \{\omega \in \mathbb{M} : \alpha\mathcal{A}\omega\}$. Consider the canonical model based on this frame defined by $\|p\| = \{\omega \in \mathbb{M} : p \in \omega\}$, for every atomic proposition p. For every formula ϕ define $\|\phi\|$ according to the semantic rules given in Section 3, with the following modified truth conditions for the operators I_t and A: $\alpha \vDash I_t\phi$ if and only if $\mathcal{I}_t(\alpha) = \|\phi\| \cap \mathcal{A}(\alpha)$ and $\alpha \vDash A\phi$ if and only if $\mathcal{A}(\alpha) \subseteq \|\phi\|$.

The crucial step in the completeness proof is the following Truth Lemma (for a proof see Goranko and Passy [12] and Bonanno [6]).

LEMMA 17. *For every $\omega \in \mathbb{M}$ and for every formula ϕ, $\omega \vDash \phi$ if and only if $\phi \in \omega$.*

With the aid of the above Lemma it can be shown that logic $\mathbb{L}$ is complete with respect to the class of augmented frames $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}, \mathcal{A}\rangle$. To complete the proof of Proposition 4, namely that logic $\mathbb{L}$ is sound and complete with respect to the class of frames $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}, \rangle$, we only need to invoke the result (Chellas, 1984, Theorem 3.12, p. 97) that completeness with respect to the class of augmented frames (where $\mathcal{A}$ is an equivalence relation) implies completeness with respect to the generated sub-frames (where $\mathcal{A}$ is the universal relation). The latter are precisely what we called frames. In a frame where the relation $\mathcal{A}$ is the universal relation the semantic rule $\alpha \vDash I_t\phi$ if and only if $\mathcal{I}_t(\alpha) = \|\phi\| \cap \mathcal{A}(\alpha)$ becomes $\alpha \vDash I_t\phi$ if and only if $\mathcal{I}_t(\alpha) = \|\phi\|$ and the semantic rule $\alpha \vDash A\phi$ if and only if $\mathcal{A}(\alpha) \subseteq \|\phi\|$ becomes $\alpha \vDash A\phi$ if and only if $\|\phi\| = \Omega$, since $\mathcal{A}(\alpha) = \Omega$.

Proof. (**Proof of Proposition 6**).

(A) Validity. Fix a frame $\langle \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}\rangle$ that satisfies QBR, that is, $\forall\omega \in \Omega, \forall t \in \mathbb{N}$, if $\mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega) \neq \varnothing$ then $\mathcal{B}_{t+1}(\omega) = \mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega)$. By Proposition 4 it is enough to show that the three axioms QA, P and M are valid in it. Fix an arbitrary model based on this frame and arbitrary state ω, date t and formulas ϕ and ψ. First we show that $\omega \models (\neg B_t\neg\phi \wedge I_{t+1}\phi) \rightarrow B_{t+1}\phi$. Suppose that $\omega \models \neg B_t\neg\phi \wedge I_{t+1}\phi$. Then $\mathcal{I}_{t+1}(\omega) = ||\phi||$ and $\mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega) \neq \varnothing$. By QBR, $\mathcal{B}_{t+1}(\omega) \subseteq \mathcal{I}_{t+1}(\omega)$ and therefore $\omega \models B_{t+1}\phi$. Next we show that $\omega \models (\neg B_t\neg\phi \wedge I_{t+1}\phi) \rightarrow (B_t\psi \rightarrow B_{t+1}\psi)$. Suppose that $\omega \models \neg B_t\neg\phi \wedge I_{t+1}\phi \wedge B_t\psi$. Then $\mathcal{I}_{t+1}(\omega) = ||\phi||$, $\mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega) \neq \varnothing$ and $\mathcal{B}_t(\omega) \subseteq ||\psi||$. By QBR, $\mathcal{B}_{t+1}(\omega) \subseteq \mathcal{B}_t(\omega)$ and therefore $\omega \models B_{t+1}\psi$. Finally we show that $\omega \models (I_{t+1}\phi \wedge B_{t+1}\psi) \rightarrow B_t(\phi \rightarrow \psi)$. Suppose that $\omega \models I_{t+1}\phi \wedge B_{t+1}\psi$. Then $\mathcal{I}_{t+1}(\omega) = ||\phi||$ and $\mathcal{B}_{t+1}(\omega) \subseteq ||\psi||$. Fix an arbitrary $\omega' \in \mathcal{B}_t(\omega)$. If $\omega' \models \neg\phi$, then $\omega' \models \phi \rightarrow \psi$; if $\omega' \models \phi$, then $\omega' \in \mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega)$ and, by QBR, $\mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega) \subseteq \mathcal{B}_{t+1}(\omega)$, so that $\omega' \models \psi$ and therefore $\omega' \models \phi \rightarrow \psi$. Hence $\omega \models B_t(\phi \rightarrow \psi)$.

(B) Completeness. Let $\mathbb{M}_{QBR}$ be the set of maximally consistent sets (MCS) of formulas of $\mathbb{L}_{QBR}$. By Proposition 4 we only need to show that the frame associated with the canonical model satisfies QBR. First we show that

$$\forall t \in \mathbb{N}, \forall\omega \in \mathbb{M}_{QBR}, \;\; if \;\; \mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega) \neq \varnothing \;\; then \;\; \mathcal{B}_{t+1}(\omega) \subseteq \mathcal{I}_{t+1}(\omega).$$

Fix an arbitrary $\alpha \in \mathbb{M}_{QBR}$ and suppose that $\mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha) \neq \varnothing$. Let $\beta \in \mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha)$. Since $\mathcal{B}_t(\alpha) \subseteq \mathbb{M}_{QBR}$, $\beta \in \mathbb{M}_{QBR}$ and therefore, by definition of $\mathcal{I}_t$, there exists a formula ϕ such that $I_{t+1}\phi \in \alpha$ and $\phi \in \beta$. Since $\beta \in \mathcal{B}_t(\alpha)$, $\neg B_t\neg\phi \in \alpha$ (see Chellas [8] Theorem 5.6, p. 172). Thus $(I_{t+1}\phi \wedge \neg B_t\neg\phi) \in \alpha$. Since QA is a theorem, $(I_{t+1}\phi \wedge \neg B_t\neg\phi) \rightarrow B_{t+1}\phi \in \alpha$. Hence $B_{t+1}\phi \in \alpha$. Fix an arbitrary $\gamma \in \mathcal{B}_{t+1}(\alpha)$. By definition of $\mathcal{B}_{t+1}$, $\{\psi : B_{t+1}\psi \in \alpha\} \subseteq \gamma$. In particular, since $B_{t+1}\phi \in \alpha$, $\phi \in \gamma$. By definition of $\mathcal{I}_{t+1}$, since $I_{t+1}\phi \in \alpha$ and $\phi \in \gamma$, $\gamma \in \mathcal{I}_{t+1}(\alpha)$.

Next we show that

$$\forall t \in \mathbb{N}, \forall\omega \in \mathbb{M}_{QBR}, \;\; if \;\; \mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega) \neq \varnothing \;\; then \;\; \mathcal{B}_{t+1}(\omega) \subseteq \mathcal{B}_t(\omega).$$

Fix an arbitrary $\alpha \in \mathbb{M}_{QBR}$ and suppose that $\mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha) \neq \varnothing$. Let $\beta \in \mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha)$. As shown above, there exists a ϕ such that $I_{t+1}\phi \in \alpha$, $\phi \in \beta$ and $\neg B_t\neg\phi \in \alpha$. By axiom P, for every formula ψ, $(I_{t+1}\phi \wedge \neg B_t\neg\phi) \rightarrow (B_t\psi \rightarrow B_{t+1}\psi) \in \alpha$. Thus

$$(B_t\psi \rightarrow B_{t+1}\psi) \in \alpha. \tag{1}$$

Fix an arbitrary $\gamma \in \mathcal{B}_{t+1}(\alpha)$. We want to show that $\gamma \in \mathcal{B}_t(\alpha)$, that is, that $\{\psi : B_t\psi \in \alpha\} \subseteq \gamma$. Let ψ be such that $B_t\psi \in \alpha$. By (1) $B_{t+1}\psi \in \alpha$ and therefore, $\psi \in \gamma$ (since $\gamma \in \mathcal{B}_{t+1}(\alpha)$ and, by definition of $\mathcal{B}_{t+1}$, $\{\psi : B_{t+1}\psi \in \alpha\} \subseteq \gamma$).

Finally we show that

$$\forall t \in \mathbb{N}, \forall \omega \in \mathbb{M}_{QBR}, \quad \mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega) \subseteq \mathcal{B}_{t+1}(\omega).$$

Fix arbitrary $\alpha, \beta \in \mathbb{M}_{QBR}$ such that $\beta \in \mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha)$. Then, as shown above, there exists a ϕ such that $I_{t+1}\phi \in \alpha$ and $\phi \in \beta$. Fix an arbitrary $\gamma \in \mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha)$. We want to show that $\gamma \in \mathcal{B}_{t+1}(\alpha)$, that is, that $\{\psi : B_{t+1}\psi \in \alpha\} \subseteq \gamma$. Let ψ be an arbitrary formula such that $B_{t+1}\psi \in \alpha$. Then $(I_{t+1}\phi \wedge B_{t+1}\psi) \in \alpha$. By axiom M, $(I_{t+1}\phi \wedge B_{t+1}\psi) \to B_t(\phi \to \psi) \in \alpha$. Thus $B_t(\phi \to \psi) \in \alpha$. Since $\gamma \in \mathcal{B}_t(\alpha)$, $(\phi \to \psi) \in \gamma$. Since $I_{t+1}\phi \in \alpha$ and $\gamma \in \mathcal{I}_t(\alpha)$, $\phi \in \gamma$. Hence $\psi \in \gamma$. ■

Proof. (**Proof of Proposition 8**). By Proposition 6, it is sufficient to show that axioms CA and WC are valid in every $WAGM$ frame and that the canonical model satisfies properties (1) and (2) of Definition 7.

Validity of axioms CA and WC. Fix an arbitrary model based on a $WAGM$ frame, a state α, a date t and a formula ϕ and suppose that $\alpha \models I_t\phi \wedge \neg A\neg\phi$. Then $\mathcal{I}_t(\alpha) = ||\phi||$ and there exists a $\beta \in \Omega$ such that $\beta \models \phi$. Thus $\mathcal{I}_t(\alpha) \neq \varnothing$ and by property (1) of Definition 7, $\mathcal{B}_t(\alpha) \subseteq \mathcal{I}_t(\alpha)$. Thus $\mathcal{B}_t(\alpha) \subseteq ||\phi||$, that is, $\alpha \models B_t\phi$ and axiom CA is valid. By property (2) of Definition 7, $\mathcal{B}_t(\alpha) \neq \varnothing$. Fix an arbitrary formula ψ and suppose that $\alpha \models B_t\psi$. Since $\mathcal{B}_t(\alpha) \neq \varnothing$, there exists a γ such that $\gamma \in \mathcal{B}_t(\alpha)$. Then $\gamma \models \psi$ and therefore $\alpha \models \neg B_t\neg\psi$. Thus axiom WC is valid.

Proof of completeness. Let $\mathbb{M}_{WAGM}$ be the set of maximally consistent sets of logic $\mathbb{L}_{WAGM}$. In order to prove that properties (1) and (2) hold we first start with the augmented canonical model $\langle \mathbb{M}_{WAGM} \cup \{\omega_0\}, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in\mathbb{N}}, \mathcal{A}\rangle$ (where $\mathcal{A}$ is an equivalence relation) and show that it satisfies the following property: $\forall \omega \in \mathbb{M}_{WAGM}$, $\forall t \in \mathbb{N}$, if $\mathcal{I}_t(\omega) \cap \mathcal{A}(\omega) \neq \varnothing$ then $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega) \cap \mathcal{A}(\omega)$ and $\mathcal{B}_t(\omega) \neq \varnothing$. Fix arbitrary $\alpha \in \mathbb{M}_{WAGM}$ and $t \in \mathbb{N}$ and suppose that $\mathcal{I}_t(\alpha) \cap \mathcal{A}(\alpha) \neq \varnothing$. Let $\beta \in \mathcal{I}_t(\alpha) \cap \mathcal{A}(\alpha)$. Since $\beta \in \mathcal{A}(\alpha)$, $\beta \in \mathbb{M}_{WAGM}$; hence (since $\beta \in \mathcal{I}_t(\alpha)$) by definition of $\mathcal{I}_t$ there exists a ϕ such that $I_t\phi \in \alpha$ and $\phi \in \beta$. Since $\beta \in \mathcal{A}(\alpha)$, $\neg A\neg\phi \in \alpha$. Thus $I_t\phi \wedge \neg A\neg\phi \in \alpha$. Since $(I_t\phi \wedge \neg A\neg\phi) \to B_t\phi$ is an axiom of $\mathbb{L}_{WAGM}$, $(I_t\phi \wedge \neg A\neg\phi) \to B_t\phi \in \alpha$. Thus $B_t\phi \in \alpha$. Fix an arbitrary $\gamma \in \mathcal{B}_t(\alpha)$. Then, by definition of $\mathcal{B}_t$, $\phi \in \gamma$ and, since $\mathcal{B}_t$ is a subrelation of $\mathcal{A}$, $\gamma \in \mathcal{A}(\alpha)$. Since $I_t\phi \in \alpha$ and $\phi \in \gamma$, by definition of $\mathcal{I}_t$, $\gamma \in \mathcal{I}_t(\alpha)$. Hence $\mathcal{B}_t(\alpha) \subseteq \mathcal{I}_t(\alpha) \cap \mathcal{A}(\alpha)$. Since $(I_t\phi \wedge \neg A\neg\phi) \to (B_t\psi \to \neg B_t\neg\psi)$ is a theorem, it belongs to α. Thus, for every formula ψ, $(B_t\psi \to \neg B_t\neg\psi) \in \alpha$. It follows from this (see Chellas [8]) that $\mathcal{B}_t(\alpha) \neq \varnothing$. As in the proof of Proposition 4, the proof is completed by taking the sub-frame generated by α. ■

PROOF. (**Proof of Proposition 9.**) (K*1). The proof that K is deductively closed (a belief set) is similar to the proof that K^*_ϕ is deductively closed, so will only prove the latter. We need to show that $K^*_\phi = \left[K^*_\phi\right]^{PL}$. Clearly, $K^*_\phi \subseteq \left[K^*_\phi\right]^{PL}$, since $\psi \to \psi$ is a tautology. Thus we only need to show that $\left[K^*_\phi\right]^{PL} \subseteq K^*_\phi$. Let $\psi \in \left[K^*_\phi\right]^{PL}$, that is, there exist $\phi_1, ..., \phi_n \in K^*_\phi$ such that $(\phi_1 \wedge ... \wedge \phi_n) \to \psi$ is a tautology. Then, by Necessitation of B_{t+1} (see Remark 3), $B_{t+1}((\phi_1 \wedge ... \wedge \phi_n) \to \psi)$ is a theorem of $\mathbb{L}_{WAGM}$ and therefore, by Proposition 8, it is valid in the given model, so that $\omega \vDash B_{t+1}\left((\phi_1 \wedge ... \wedge \phi_n) \to \psi\right)$. By definition of K^*_ϕ, since $\phi_1, ..., \phi_n \in K^*_\phi$, $\omega \vDash B_{t+1}\left(\phi_1 \wedge ... \wedge \phi_n\right)$. By axiom K_B, $\omega \vDash B_{t+1}\left((\phi_1 \wedge ... \wedge \phi_n) \to \psi\right) \wedge B_{t+1}\left(\phi_1 \wedge ... \wedge \phi_n\right) \to B_{t+1}\psi$. Thus $\omega \vDash B_{t+1}\psi$, that is, $\psi \in K^*_\phi$.

(K*2). By hypothesis, there exists an $\omega' \in \Omega$ such that $\omega' \vDash \phi$. Hence $\omega \vDash \neg A\neg\phi$. Since $(I_{t+1}\phi \wedge \neg A\neg\phi) \to B_{t+1}\phi$ is an axiom of $\mathbb{L}_{WAGM}$, it is valid in the given model and, therefore, $\omega \vDash (I_{t+1}\phi \wedge \neg A\neg\phi) \to B_{t+1}\phi$. Thus, since, by hypothesis, $\omega \vDash I_{t+1}\phi$, $\omega \vDash B_{t+1}\phi$, that is, $\phi \in K^*_\phi$.

(K*3). Let $\psi \in K^*_\phi$, i.e. $\omega \vDash B_{t+1}\psi$. By axiom M, $\omega \vDash (I_{t+1}\phi \wedge B_{t+1}\psi) \to B_t(\phi \to \psi)$. By hypothesis, $\omega \vDash I_{t+1}\phi$. Thus $\omega \vDash B_t(\phi \to \psi)$, that is, $(\phi \to \psi) \in K$. Hence $\{\phi, (\phi \to \psi)\} \in K \cup \{\phi\}$ and, since $(\phi \wedge (\phi \to \psi)) \to \psi$ is a tautology, $\psi \in [K \cup \{\phi\}]^{PL}$.

(K*4). Suppose $\neg\phi \notin K$, that is, $\omega \vDash \neg B_t\neg\phi$. By axiom P, for every formula ψ, $\omega \vDash (I_{t+1}\phi \wedge \neg B_t\neg\phi) \to (B_t\psi \to B_{t+1}\psi)$. Thus, since by hypothesis $\omega \vDash I_{t+1}\phi$,

$$\omega \vDash (B_t\psi \to B_{t+1}\psi) \text{ for every formula } \psi. \tag{2}$$

Let $\chi \in [K \cup \{\phi\}]^{PL}$, that is, there exist $\phi_1, ..., \phi_n \in K \cup \{\phi\}$ such that $(\phi_1 \wedge ... \wedge \phi_n) \to \chi$ is a tautology. We want to show that $\chi \in K^*_\phi$, i.e. $\omega \vDash B_{t+1}\chi$. Since $(\phi_1 \wedge ... \wedge \phi_n) \to \chi$ is a tautology, $\omega \vDash B_t\left((\phi_1 \wedge ... \wedge \phi_n) \to \chi\right)$. If $\phi_1, ..., \phi_n \in K$, then $\omega \vDash B_t\left(\phi_1 \wedge ... \wedge \phi_n\right)$ and therefore (by axiom K_B) $\omega \vDash B_t\chi$. Thus, by (2), $\omega \vDash B_{t+1}\chi$. If it is not the case that $\phi_i \in K$ for all $i = 1, ..., n$, then, by renumbering the formulas if necessary, we can assume that $\phi_1 = \phi$ and $\phi_2, ..., \phi_n \in K$. In this case we have $\omega \vDash B_t\left(\phi_2 \wedge ... \wedge \phi_n\right)$ and $\omega \vDash B_t\left((\phi_2 \wedge ... \wedge \phi_n) \to (\phi \to \chi)\right)$ since $(\phi \wedge \phi_2 \wedge ... \wedge \phi_n) \to \chi$ is a tautology and it is equivalent to $(\phi_2 \wedge ... \wedge \phi_n) \to (\phi \to \chi)$. Thus (by axiom

K_B) $\omega \vDash B_t(\phi \to \chi)$. Hence, by (2) (with $\psi = (\phi \to \chi)$), $\omega \vDash B_{t+1}(\phi \to \chi)$. By axiom CA, $\omega \vDash (I_{t+1}\phi \wedge \neg A\neg\phi) \to B_{t+1}\phi$. By hypothesis $\omega \vDash I_{t+1}\phi \wedge \neg A\neg\phi$. Hence $\omega \vDash B_{t+1}\phi$. If follows from axiom K_B that $\omega \vDash B_{t+1}\chi$.

(K*5). We have to show that $K^*_\phi \neq \Gamma$ (since, by hypothesis, we have ruled out the possibility that ϕ is a contradiction). By (K*1) $K^*_\phi = \left[K^*_\phi\right]^{PL}$. By hypothesis, $\omega \vDash I_{t+1}\phi \wedge \neg A\neg\phi$ and therefore, by axiom WC, for every formula ψ, $\omega \vDash B_{t+1}\psi \to \neg B_{t+1}\neg\psi$. Thus, since $\omega \vDash B_{t+1}(p \vee \neg p)$ (because $(p \vee \neg p)$ is a tautology), $\omega \vDash \neg B_{t+1}\neg(p \vee \neg p)$, so that $\neg(p \vee \neg p) \notin K^*_\phi$ and hence $K^*_\phi \neq \Gamma$.

(K*6). We have to show that if $\phi \leftrightarrow \psi$ is a tautology then $K^*_\phi = K^*_\psi$. If $\phi \leftrightarrow \psi$ is a tautology, then $\|\phi \leftrightarrow \psi\| = \Omega$, that is, $\|\phi\| = \|\psi\|$. Thus $\omega \vDash I_{t+1}\phi$ if and only if $\omega \vDash I_{t+1}\psi$. Hence, by definition of K^*_ϕ, $K^*_\phi = K^*_\psi$. ■

In order to prove Proposition 12 we need some preliminary results.

LEMMA 18. *Let $\precsim$ be a complete and transitive binary relation on Ω and $X \subseteq Y \subseteq \Omega$. If $(\min_\precsim Y) \cap X \neq \varnothing$ then $\min_\precsim X = (\min_\precsim Y) \cap X$.*

PROOF. First we show that $(\min_\precsim Y) \cap X \subseteq \min_\precsim X$. Let $\beta \in (\min_\precsim Y) \cap X$. Then $\beta \in X$ and $\beta \precsim \gamma$ for all $\gamma \in Y$. Since $X \subseteq Y$, it follows that $\beta \in \min_\precsim X$. Next we show that if $(\min_\precsim Y) \cap X \neq \varnothing$ then $\min_\precsim X \subseteq (\min_\precsim Y) \cap X$. Let $\beta \in (\min_\precsim Y) \cap X$. Fix an arbitrary $\gamma \in \min_\precsim X$. Then $\gamma \in X$ and $\gamma \precsim \beta$. Suppose that $\gamma \notin \min_\precsim Y$. Then there exists a $\delta \in Y$ such that $\delta \prec \gamma$ (that is, $\delta \precsim \gamma$ and $\gamma \not\precsim \delta$). By transitivity (since $\gamma \precsim \beta$), $\delta \prec \beta$, contradicting the fact that $\beta \in \min_\precsim Y$. ■

LEMMA 19. *Let $\{\mathcal{B}_t(\alpha), \mathcal{I}_t(\alpha)\}_{t\in\mathbb{N}}$ be a belief revision history that satisfies information refinement (that is, $\forall t \in \mathbb{N}$, $\mathcal{I}_{t+1}(\alpha) \subseteq \mathcal{I}_t(\alpha)$). Let $t_0, t_1 \in \mathbb{N}$ with $t_0 < t_1$ and suppose that, $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_1}(\alpha) \neq \varnothing$. Then, for every $t \in \mathbb{N}$ with $t_0 < t \leq t_1$, $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_t(\alpha) \neq \varnothing$.*

PROOF. Fix a t such that $t_0 < t \leq t_1$. By information refinement, $\mathcal{I}_{t_1}(\alpha) \subseteq \mathcal{I}_{t_1-1}(\alpha) \subseteq ... \subseteq \mathcal{I}_t(\alpha)$. Thus $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_1}(\alpha) \subseteq \mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_t(\alpha)$. By hypothesis, $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_1}(\alpha) \neq \varnothing$. ■

LEMMA 20. *Let $\{\mathcal{B}_t(\alpha), \mathcal{I}_t(\alpha)\}_{t\in\mathbb{N}}$ be a belief revision history which (1) is Qualitatively Bayesian, that is, $\forall t \in \mathbb{N}$, if $\mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha) \neq \varnothing$ then $\mathcal{B}_{t+1}(\alpha) = \mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha)$ and (2) satisfies information refinement, that is, $\forall t \in \mathbb{N}$, $\mathcal{I}_{t+1}(\alpha) \subseteq \mathcal{I}_t(\alpha)$. Let $T_0, T_1 \in \mathbb{N}$ be such that $T_0 < T_1$ and, $\forall t \in \mathbb{N}$ with $T_0 < t \leq T_1$, $\mathcal{B}_{T_0}(\alpha) \cap \mathcal{I}_t(\alpha) \neq \varnothing$. Then, $\forall t \in \mathbb{N}$ with $T_0 < t \leq T_1$, $\mathcal{B}_t(\alpha) = \mathcal{B}_{T_0}(\alpha) \cap \mathcal{I}_t(\alpha)$.*

PROOF. We prove it by induction. The statement is clearly true for $t = T_0 + 1$, since by hypothesis $\mathcal{B}_{T_0}(\alpha) \cap \mathcal{I}_{T_0+1}(\alpha) \neq \varnothing$ and by the Qualitative Bayes Rule (QBR) $\mathcal{B}_{T_0+1}(\alpha) = \mathcal{B}_{T_0}(\alpha) \cap \mathcal{I}_{T_0+1}(\alpha)$. If $T_1 = T_0 + 1$ there is nothing else to prove. Suppose therefore that $T_1 > T_0 + 1$ and proceed with the induction step: suppose that the statement is true for every $t \in \mathbb{N}$ with $T_0 < t \leq T$ (with $T < T_1$). We want to show that it is true for $t = T + 1$. By the induction hypothesis, $\mathcal{B}_T(\alpha) = \mathcal{B}_{T_0}(\alpha) \cap \mathcal{I}_T(\alpha)$. Thus $\mathcal{B}_T(\alpha) \cap \mathcal{I}_{T+1}(\alpha) = \mathcal{B}_{T_0}(\alpha) \cap \mathcal{I}_T(\alpha) \cap \mathcal{I}_{T+1}(\alpha)$. Since, by hypothesis, $\mathcal{I}_{T+1}(\alpha) \subseteq \mathcal{I}_T(\alpha)$, $\mathcal{I}_{T+1}(\alpha) \cap \mathcal{I}_T(\alpha) = \mathcal{I}_{T+1}(\alpha)$. Thus

$$\mathcal{B}_T(\alpha) \cap \mathcal{I}_{T+1}(\alpha) = \mathcal{B}_{T_0}(\alpha) \cap \mathcal{I}_{T+1}(\alpha). \tag{3}$$

By hypothesis, $\mathcal{B}_{T_0}(\alpha) \cap \mathcal{I}_{T+1}(\alpha) \neq \varnothing$. Hence, by (3),

$$\mathcal{B}_T(\alpha) \cap \mathcal{I}_{T+1}(\alpha) \neq \varnothing. \tag{4}$$

It follows from QBR that

$$\mathcal{B}_{T+1}(\alpha) = \mathcal{B}_T(\alpha) \cap \mathcal{I}_{T+1}(\alpha). \tag{5}$$

From (3) and (5) we get that $\mathcal{B}_{T+1}(\alpha) = \mathcal{B}_{T_0}(\alpha) \cap \mathcal{I}_{T+1}(\alpha)$. ∎

LEMMA 21. *Let $\{\mathcal{B}_t(\alpha), \mathcal{I}_t(\alpha)\}_{t\in\mathbb{N}}$ be a belief revision history which is Qualitatively Bayesian and satisfies information refinement. Let $t_0, t_1 \in \mathbb{N}$ with $t_0 < t_1$ and suppose that, $\forall t \in \mathbb{N}$ with $t_0 < t \leq t_1$, $\mathcal{B}_{t-1}(\alpha) \cap \mathcal{I}_t(\alpha) \neq \varnothing$. Then, $\forall t \in \mathbb{N}$ with $t_0 < t \leq t_1$, $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_t(\alpha) \neq \varnothing$.*

PROOF. We prove it by induction. The statement is true for $t = t_0 + 1$ since, by hypothesis, $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_0+1}(\alpha) \neq \varnothing$. Now the induction step. Let $T < t_1$ and suppose that the statement is true for all t up to T, that is,

$$\forall t \in \mathbb{N} \text{ with } t_0 < t \leq T,\ \ \mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_t(\alpha) \neq \varnothing. \tag{6}$$

We want to show that it is true for $t = T+1$, that is, that $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{T+1}(\alpha) \neq \varnothing$. By (6) and Lemma 20 (with $T_0 = t_0$ and $T_1 = T$),

$$\mathcal{B}_T(\alpha) = \mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_T(\alpha). \tag{7}$$

Thus

$$\mathcal{B}_T(\alpha) \cap \mathcal{I}_{T+1}(\alpha) = \mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_T(\alpha) \cap \mathcal{I}_{T+1}(\alpha). \tag{8}$$

Since, by hypothesis, $\mathcal{I}_{T+1}(\alpha) \subseteq \mathcal{I}_T(\alpha)$, $\mathcal{I}_T(\alpha) \cap \mathcal{I}_{T+1}(\alpha) = \mathcal{I}_{T+1}(\alpha)$. It follows from this and (8) that

$$\mathcal{B}_T(\alpha) \cap \mathcal{I}_{T+1}(\alpha) = \mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{T+1}(\alpha). \tag{9}$$

Since, by hypothesis, $\mathcal{B}_T(\alpha) \cap \mathcal{I}_{T+1}(\alpha) \neq \varnothing$, it follows from (9) that $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{T+1}(\alpha) \neq \varnothing$. ∎

COROLLARY 22. *Let $\{\mathcal{B}_t(\alpha), \mathcal{I}_t(\alpha)\}_{t\in\mathbb{N}}$ be a belief revision history which is Qualitatively Bayesian and satisfies information refinement. Let $t_0, t_1 \in \mathbb{N}$ with $t_0 < t_1$ and suppose that $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_1}(\alpha) = \varnothing$. Then there exists a $t \in \mathbb{N}$ with $t_0 < t \leq t_1$ such that $\mathcal{B}_{t-1}(\alpha) \cap \mathcal{I}_t(\alpha) = \varnothing$.*

PROOF. If not, then, for every t with $t_0 < t \leq t_1$, $\mathcal{B}_{t-1}(\alpha) \cap \mathcal{I}_t(\alpha) \neq \varnothing$ and by Lemma 21 $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_1}(\alpha) \neq \varnothing$, yielding a contradiction. ∎

LEMMA 23. *$\forall t, t' \in \mathbb{N}$, $\forall \omega' \in \Omega$, if $\omega' \in \mathcal{I}_t(\alpha) \backslash \mathcal{B}_t(\alpha)$ and $\omega' \in \mathcal{B}_{t'}(\alpha)$ then $t' > t$ and $\mathcal{B}_t(\alpha) \cap \mathcal{I}_{t'}(\alpha) = \varnothing$.*

PROOF. First we prove that $t' > t$. Let $\omega' \in \mathcal{I}_t(\alpha) \backslash \mathcal{B}_t(\alpha)$ and $\omega' \in \mathcal{B}_{t'}(\alpha)$. Then $\omega' \in \mathcal{B}_{t'}(\alpha) \cap \mathcal{I}_t(\alpha)$, so that $t \neq t'$ and $\mathcal{B}_{t'}(\alpha) \cap \mathcal{I}_t(\alpha) \neq \varnothing$. If $t' < t$, then by Lemmas 19 and 20, $\mathcal{B}_t(\alpha) = \mathcal{B}_{t'}(\alpha) \cap \mathcal{I}_t(\alpha)$, so that $\omega' \in \mathcal{B}_t(\alpha)$, contradicting the hypothesis that $\omega' \in \mathcal{I}_t(\alpha) \backslash \mathcal{B}_t(\alpha)$. Thus $t' > t$. Next we prove that $\mathcal{B}_t(\alpha) \cap \mathcal{I}_{t'}(\alpha) = \varnothing$. Suppose that $\mathcal{B}_t(\alpha) \cap \mathcal{I}_{t'}(\alpha) \neq \varnothing$. Then by Lemmas 19 and 20 $\mathcal{B}_{t'}(\alpha) = \mathcal{B}_t(\alpha) \cap \mathcal{I}_{t'}(\alpha)$, so that $\mathcal{B}_{t'}(\alpha) \subseteq \mathcal{B}_t(\alpha)$, contradicting the hypothesis that $\omega' \in \mathcal{B}_{t'}(\alpha)$ and $\omega' \in \mathcal{I}_t(\alpha) \backslash \mathcal{B}_t(\alpha)$. ∎

PROOF. **Proof of Proposition 12**.
First we prove that if $\{\mathcal{B}_t(\alpha), \mathcal{I}_t(\alpha)\}_{t\in\mathbb{N}}$ is information-refined ($\forall t \in \mathbb{N}$, $\mathcal{I}_{t+1}(\alpha) \subseteq \mathcal{I}_t(\alpha)$) and $\precsim$ rationalizes $\{\mathcal{B}_t(\alpha), \mathcal{I}_t(\alpha)\}_{t\in\mathbb{N}}$ then $\{\mathcal{B}_t(\alpha), \mathcal{I}_t(\alpha)\}_{t\in\mathbb{N}}$ is Qualitatively Bayesian. Fix an arbitrary $t \in \mathbb{N}$ such that $\mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha) \neq \varnothing$. By hypothesis, $\mathcal{B}_t(\alpha) = \min_{\precsim} \mathcal{I}_t(\alpha)$ and $\mathcal{B}_{t+1}(\alpha) = \min_{\precsim} \mathcal{I}_{t+1}(\alpha)$. Since $\mathcal{I}_{t+1}(\alpha) \subseteq \mathcal{I}_t(\alpha)$ it follows from Lemma 18 (with $X = \mathcal{I}_{t+1}(\alpha)$ and $Y = \mathcal{I}_t(\alpha)$) that $\mathcal{B}_{t+1}(\alpha) = \mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha)$.

Next we prove that if $\{\mathcal{B}_t(\alpha), \mathcal{I}_t(\alpha)\}_{t\in\mathbb{N}}$ is consistent, successful, information-refined and Qualitatively Bayesian then there exists a plausibility relation $\precsim$ on Ω that rationalizes it. Define the function $rank : \Omega \to \mathbb{N}$ as follows:

$$
\begin{aligned}
rank(\omega) \quad &= 0 \quad && \text{if } \omega \in \mathcal{B}_0(\alpha) \\
&= \infty && \text{if } \omega \in \Omega \setminus \textstyle\bigcup_{t\in\mathbb{N}} \mathcal{B}_t(\alpha) \\
&= t && \text{if } \omega \in \mathcal{B}_t(\alpha) \text{ and } \mathcal{B}_{t-1}(\alpha) \cap \mathcal{I}_t(\alpha) = \varnothing.
\end{aligned}
$$

First we show that this function's domain is indeed Ω. Fix arbitrary $t' \in \mathbb{N}\backslash\{0\}$ and $\omega \in \mathcal{B}_{t'}(\alpha)$. If $\mathcal{B}_{t'-1}(\alpha) \cap \mathcal{I}_{t'}(\alpha) = \varnothing$ then $rank(\omega) = t'$. If $\mathcal{B}_{t'-1}(\alpha) \cap \mathcal{I}_{t'}(\alpha) \neq \varnothing$ let $T = \{t \in \mathbb{N} : t < t' \text{ and } \mathcal{B}_{t-1}(\alpha) \cap \mathcal{I}_t(\alpha) = \varnothing\}$. If $T = \varnothing$, then, by Lemma 21 (with $t_0 = 0$ and $t_1 = t'$), for every $t \in \mathbb{N}$ with $0 < t \leq t'$, $\mathcal{B}_0(\alpha) \cap \mathcal{I}_t(\alpha) \neq \varnothing$. It follows from this and Lemma 20 that $\mathcal{B}_{t'-1}(\alpha) \cap \mathcal{I}_{t'}(\alpha) = \mathcal{B}_0(\alpha) \cap \mathcal{I}_{t'}(\alpha)$. By the Qualitative Bayes Rule (since, by hypothesis, $\mathcal{B}_{t'-1}(\alpha) \cap \mathcal{I}_{t'}(\alpha) \neq \varnothing$), $\mathcal{B}_{t'}(\alpha) = \mathcal{B}_{t'-1}(\alpha) \cap \mathcal{I}_{t'}(\alpha)$.

Thus $\mathcal{B}_{t'}(\alpha) = \mathcal{B}_0(\alpha) \cap \mathcal{I}_{t'}(\alpha)$ and therefore $\omega \in \mathcal{B}_0(\alpha)$, so that $rank(\omega) = 0$. If $T \neq \varnothing$, let $t_{\max} = \max T$. Then

$$\mathcal{B}_{t_{\max}-1}(\alpha) \cap \mathcal{I}_{t_{\max}}(\alpha) = \varnothing. \tag{10}$$

Since, by hypothesis, $\mathcal{B}_{t'-1}(\alpha) \cap \mathcal{I}_{t'}(\alpha) \neq \varnothing$ it follows from (10) that $t_{\max} < t'$. Furthermore, by definition of $t_{\max}$,

$$\forall t \in \mathbb{N} \text{ with } t_{\max} < t \leq t',\ \ \mathcal{B}_{t-1}(\alpha) \cap \mathcal{I}_t(\alpha) \neq \varnothing. \tag{11}$$

By Lemma 21 (with $t_0 = t_{\max}$ and $t_1 = t'$), for every t such that $t_{\max} < t \leq t'$, $\mathcal{B}_{t_{\max}}(\alpha) \cap \mathcal{I}_t(\alpha) \neq \varnothing$. It follows from this and Lemma 20 that $\mathcal{B}_{t'-1}(\alpha) \cap \mathcal{I}_{t'}(\alpha) = \mathcal{B}_{t_{\max}}(\alpha) \cap \mathcal{I}_{t'}(\alpha)$. By the Qualitative Bayes Rule (since, by hypothesis, $\mathcal{B}_{t'-1}(\alpha) \cap \mathcal{I}_{t'}(\alpha) \neq \varnothing$), $\mathcal{B}_{t'}(\alpha) = \mathcal{B}_{t'-1}(\alpha) \cap \mathcal{I}_{t'}(\alpha)$. Thus $\mathcal{B}_{t'}(\alpha) = \mathcal{B}_{t_{\max}}(\alpha) \cap \mathcal{I}_{t'}(\alpha)$ and therefore $\omega \in \mathcal{B}_{t_{\max}}(\alpha)$, so that, by (10) and the definition of the function *rank*, $rank(\omega) = t_{\max}$.

Now define the binary relation $\precsim$ on Ω as follows: $\omega \precsim \omega'$ if and only if $rank(\omega) \leq rank(\omega')$ (with the convention that $n < \infty$ for every $n \in \mathbb{N}$). Clearly $\precsim$ is complete and transitive. Now we show that $\precsim$ rationalizes $\{\mathcal{B}_t(\alpha), \mathcal{I}_t(\alpha)\}_{t\in\mathbb{N}}$. Fix an arbitrary t. We want to show that $\mathcal{B}_t(\alpha) = \min_{\precsim} \mathcal{I}_t(\alpha)$, that is, (1) for every $\omega \in \mathcal{B}_t(\alpha)$ and for every $\omega' \in \mathcal{I}_t(\alpha)$, $rank(\omega) \leq rank(\omega')$ and (2) if $\omega \in \mathcal{I}_t(\alpha)$ is such that, for every $\omega' \in \mathcal{I}_t(\alpha)$, $rank(\omega) \leq rank(\omega')$, then $\omega \in \mathcal{B}_t(\alpha)$. Recall that, by hypothesis the belief history is consistent and successful, so that, for every t, $\varnothing \neq \mathcal{B}_t(\alpha) \subseteq \mathcal{I}_t(\alpha)$.

The proof is by induction. The statement is true for $t = 0$ since (1) $\varnothing \neq \mathcal{B}_0(\alpha) \subseteq \mathcal{I}_0(\alpha)$ and (2) by construction $rank(\omega) \geq 0$ for every $\omega \in \Omega$ and $rank(\omega) = 0$ if and only if $\omega \in \mathcal{B}_0(\alpha)$. Now let $t_0 \geq 0$ and suppose that the statement is true for every $t \leq t_0$, that is, for every such t, $\mathcal{B}_t(\alpha) = \min_{\precsim} \mathcal{I}_t(\alpha)$. Then for every $t \leq t_0$ there exists an $n_t \in \mathbb{N}$ such that

$$\forall \omega \in \mathcal{B}_t(\alpha),\ rank(\omega) = n_t \text{ and } \forall \omega \in \mathcal{I}_t(\alpha) \backslash \mathcal{B}_t(\alpha),\ rank(\omega) > n_t. \tag{12}$$

We want to show that the same is true for $t = t_0 + 1$. We need to consider two cases.

CASE 1: $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_0+1}(\alpha) \neq \varnothing$. By (12) there exists an $n_{t_0} \in \mathbb{N}$ such that $\forall \omega \in \mathcal{B}_{t_0}(\alpha)$, $rank(\omega) = n_{t_0}$ and $\forall \omega \in \mathcal{I}_{t_0}(\alpha) \backslash \mathcal{B}_{t_0}(\alpha)$, $rank(\omega) > n_{t_0}$. Since $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_0+1}(\alpha) \neq \varnothing$, by the Qualitative Bayes Rule, $\mathcal{B}_{t_0+1}(\alpha) = \mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_0+1}(\alpha)$ it follows from this and the fact that $\mathcal{I}_{t_0+1}(\alpha) \subseteq \mathcal{I}_{t_0}(\alpha)$, that (1) $\forall \omega \in \mathcal{B}_{t_0+1}(\alpha)$, $rank(\omega) = n_{t_0}$ and (2) $\forall \omega \in \mathcal{I}_{t_0+1}(\alpha) \backslash \mathcal{B}_{t_0+1}(\alpha)$, $rank(\omega) > n_{t_0}$.[11] Thus $\mathcal{B}_{t_0+1}(\alpha) = \min_{\precsim} \mathcal{I}_{t_0+1}(\alpha)$.

[11] Let $\omega \in \mathcal{I}_{t_0+1}(\alpha) \backslash \mathcal{B}_{t_0+1}(\alpha)$. Since $\mathcal{I}_{t_0+1}(\alpha) \subseteq \mathcal{I}_{t_0}(\alpha)$, $\omega \in \mathcal{I}_{t_0}(\alpha)$. If $\omega \notin \mathcal{B}_{t_0}(\alpha)$, then $rank(\omega) > n_{t_0}$. If $\omega \in \mathcal{B}_{t_0}(\alpha)$ then $\omega \in \mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_0+1}(\alpha)$ which, by *QBR*, implies that $\omega \in \mathcal{B}_{t_0+1}(\alpha)$, contradicting the hypothesis that $\omega \in \mathcal{I}_{t_0+1}(\alpha) \backslash \mathcal{B}_{t_0+1}(\alpha)$.

CASE 2: $\mathcal{B}_{t_0}(\alpha) \cap \mathcal{I}_{t_0+1}(\alpha) = \varnothing$. By definition of $rank$, $\forall \omega \in \mathcal{B}_{t_0+1}(\alpha)$, $rank(\omega) = t_0 + 1$. We need to show that if $\omega' \in \mathcal{I}_{t_0+1}(\alpha) \backslash \mathcal{B}_{t_0+1}(\alpha)$ then $rank(\omega') > t_0 + 1$. If $\omega' \in \Omega \setminus \bigcup_{t \in \mathbb{N}} \mathcal{B}_t(\alpha)$ then, by definition of $rank$, $rank(\omega') = \infty$. Suppose, therefore, that $\omega' \in \mathcal{B}_{t'}(\alpha)$ for some t'. By Lemma 23, $t' > t_0 + 1$ and $\mathcal{B}_{t_0+1}(\alpha) \cap \mathcal{I}_{t'} = \varnothing$. Hence, by Corollary 22 there exists a $t_1 \in \mathbb{N}$ with $t_0 + 1 < t_1 \leq t'$ such that $\mathcal{B}_{t_1-1}(\alpha) \cap \mathcal{I}_{t_1}(\alpha) = \varnothing$. If $\omega' \in \mathcal{B}_{t_1}(\alpha)$ then, by definition of $rank$, $rank(\omega') = t_1 > t_0 + 1$. If $\omega' \notin \mathcal{B}_{t_1}(\alpha)$ then $t_1 \neq t'$ and therefore $t_1 < t'$. By Corollary 22 it follows from this and the fact that $\mathcal{B}_{t_1-1}(\alpha) \cap \mathcal{I}_{t_1}(\alpha) = \varnothing$, that there exists a $t_2 \in \mathbb{N}$ with $t_1 < t_2 \leq t'$ such that $\mathcal{B}_{t_2-1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) = \varnothing$. If $\omega' \in \mathcal{B}_{t_2}(\alpha)$ then, by definition of $rank$, $rank(\omega') = t_2$, otherwise a finite repetition of this argument yields that $rank(\omega') = t$ for some t such that $t \geq t_1 > t_0 + 1$. ∎

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